Innovativity in Modeling and Analytics Journal of Research
vol. 5, 2020, pp. 50-55,
http://imajor.info/

ISSN 2534-9619

The Fast Decoupled Algorithm for a Nonsymmetric Nash-Riccati Equation

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Abstract. We investigate a nonsymmetric Nash-Riccati equation. The stabilising solution of the nonsymmetric Nash-Riccati equation is applied for computing the equilibrium point in linear quadratic games for positive systems. We consider a decoupled modification of the linearized Newton method for computing the stabilizing solution. We provide numerical experiments with the new iteration and compare the results with the classical linearized Newton method.

Key Words: game models, nonsymmetric Nash-Riccati equation, decoupled iteration.

1 Introduction

We investigate the nonsymmetric matrix Riccati equation in the special form:

$$\mathfrak{R}(\mathcal{X}) = -\mathcal{D}\mathcal{X} - \mathcal{X}A + \mathcal{X}\mathcal{S}\mathcal{X} - \mathcal{Q} = 0 \,. \tag{1}$$

The unknown matrix $\mathcal{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is not square. For this reason the matrix coefficients have different dimensions, i.e. they are : $(-A)$ is an $n \times n$ M-matrix, $\mathcal{D} = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}$, $\mathcal{S} = (S_1 \quad S_2)$ where $S_i = B_i\, R_{ii}^{-1}\, B_i^T$ is an $n \times n$ symmetric nonpositive matrix, $i = 1, 2$, $\mathcal{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ and Q_i is an $n \times n$ symmetric nonnegative matrix.

We rewrite the matrix function $\mathfrak{R}(\mathcal{X})$ in the form $\mathfrak{R}(\mathcal{X}) = \begin{pmatrix} \mathcal{R}_1(X_1, X_2) \\ \mathcal{R}_2(X_1, X_2) \end{pmatrix}$, where

$$\mathcal{R}_1(X_1, X_2) = -A^T X_1 - X_1 A + X_1 S_1 X_1 + X_1 S_2 X_2 - Q_1 \,,$$
$$\mathcal{R}_2(X_1, X_2) = -A^T X_2 - X_2 A + X_2 S_1 X_1 + X_2 S_2 X_2 - Q_2 \,.$$

The equation $\mathfrak{R}(\mathcal{X}) = 0$ is equivalent to the set of Riccati equations $\mathcal{R}_1(X_1, X_2) = 0$, $\mathcal{R}_2(X_1, X_2) = 0$.

The notation $\mathbf{R}^{s \times q}$ stands for $s \times q$ real matrices. In this investigation we exploit the properties of nonnegative matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{m \times n}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We use an elementwise order relation. The inequality $P \geq Q(P > Q)$ for $P = (p_{ij}), Q = (q_{ij})$ means that $p_{ij} \geq q_{ij}(p_{ij} > q_{ij})$ for all indexes i and j. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$.

The computation of the minimal nonnegative solution to (**??**) with a nonsingular M-matrix is a popular subject in recent years. Several methods from the family of linearized implicit iterations were proposed and investigated in [1, 2, 5].

In [3], authors have proposed the following Alternately Linearized Implicit Decoupled Iteration (ALIDI) for computing the stabilizing solution of (1):

$$Y_i^{(k)}(\mu I_n + A - S_1X_1^{(k)} - S_2X_2^{(k)}) = (\mu I - A^T)X_i^{(k)} - Q_i\,, \quad i = 1,2, \tag{2}$$

$$(\mu I_n + A^T - Y_i^{(k)}S_i)X_i^{(k+1)} = Y_i^{(k)}(\mu I - A + S_jX_j^{(k)}) - Q_i\,, \quad i,j = 1,2,\ j \neq i\,, \tag{3}$$

$X_1^{(0)} = X_2^{(0)} = 0\,, \quad k = 0,1,2,\ldots\,,\ \gamma < 0\,.$

The convergence properties of (2)-(3) are derived in [3]. The following theorem is proved.

Theorem 1. (Theorem 2,[3]) Assume the matrix -A is an M-matrix and $\mathcal{Q} \geq 0, \mathcal{S} \leq 0\,, \gamma < 0$, such that $(-\gamma - A)$ is an M-matrix and $(\gamma - A)$ is nonpositive. Assume there exist symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$, such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0, i = 1,2$ and $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ is an M-matrix. The matrix sequences $\{X_1^{(k)}, X_1^{(k)}\}_{k=0}^{\infty}$ defined by (2)-(3) satisfy the following properties:

(i) $\hat{X}_i, \geq X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ for $i = 1,2\,, \quad k = 0,1,\ldots;$

(ii) $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0\,, \quad \mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}) \leq 0\,, \quad \mathcal{R}_i(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0\,,$
$i = 1,2\,, \quad k = 0,1,\ldots\,.$

(iii) The matrix sequences $\{X_1^{(k)}\,, X_2^{(k)}\}_{k=0}^{\infty}$ converge to the nonnegative minimal solution $\tilde{X}_1, \tilde{X}_2$ to the set of Riccati equations $\mathcal{R}_1(X_1, X_2)) = 0\,, \mathcal{R}_2(X_1, X_2)) = 0$ with $\tilde{X}_i \leq \hat{X}_i$ and the matrix $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ is asymptotically stable.

Here, we slightly modify the above iteration. The modification is based on the transformation $\mu I_n + A - S_iX_i^{(k)} - S_jX_j^{(k)} = L_i^{(k)} - U_i^{(k)}\,, i,j = 1,2, j \neq i,\ k = 0,1,2,\ldots$, where $L_i^{(k)}$ is the lower triangular part of the given matrix and $U_i^{(k)}$ is the strictly upper triangular part. We call it Decoupled Iteration 1 (DI1). It has the form:

$$Y_i^{(k)}\,L_i^{(k)} = (\mu I - A^T)X_i^{(k)} + X_i^{(k)}U_i^{(k)} - Q_i\,, \tag{4}$$

$$(\mu I_n + A^T)X_i^{(k+1)} = Y_i^{(k)}(\mu I - A + S_iY_i^{(k)} + S_jY_j^{(k)}) - Q_i\,, \tag{5}$$

$i,j = 1,2, j \neq i\ X_i^{(0)} = 0\,, i = 1,2 \quad k = 0,1,2,\ldots\,,\ \mu < 0\,.$

2 Convergence properties (4)-(5)

We derive some properties and identities of the matrix functions $\mathcal{R}_1(.)$, $\mathcal{R}_2(.)$.

Lemma 1 *We construct the matrix sequences $\{X_i{}^{(k)}, Y_i{}^{(k)}\}_{k=0}^{\infty}$ using (4)-(5) with initial values $X_i^{(0)} = 0\,, i = 1, 2$. The following properties hold ($i, j = 1, 2, j \neq i$):*

(i) $\mathcal{R}_i(X_1{}^{(k)}, X_2{}^{(k)}) \quad = \; (Y_i^{(k)} - X_i^{(k)})L_i^{(k)}\,,$

(ii) $\mathcal{R}_i(Y_1{}^{(k)}, Y_2{}^{(k)}) = \; (\mu I_n - A^T + Y_i^{(k)} S_i)(Y_i^{(k)} - X_i^{(k)}) + (Y_i^{(k)} - X_i^{(k)})U_i^{(k)}$
$+Y_i^{(k)} S_j(Y_j^{(k)} - X_j^{(k)})\,,$

(iii) $\mathcal{R}_i(Y_1{}^{(k)}, Y_2{}^{(k)}) = (\mu I_n + A^T)(X_i^{(k+1)} - Y_i^{(k)})\,,$

(iv) $\mathcal{R}_i(X_1{}^{(k+1)}, X_2{}^{(k+1)}) \quad = (X_i^{(k+1)} - Y_i^{(k)})(\mu I_n - A + S_i Y_i^{(k)} + S_j Y_j^{(k)})$
$+X_i^{(k+1)} S_i(X_i^{(k+1)} - Y_i^{(k)}) + X_i^{(k+1)} S_j(X_j^{(k+1)} - Y_j^{(k)})\,.$

In addition, the following equalities are true for any nonnegative matrices $\hat{X}_1, \hat{X}_2$:

(v)
$$\begin{aligned}\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \;&=\; (Y_i^{(k)} - \hat{X}_i)L_i^{(k)} + (\mu I_n - A^T + \hat{X}_i S_i)(\hat{X}_i - X_i^{(k)})\\ &\quad +(\hat{X}_i - X_i^{(k)})\,U_i^{(k)} + \hat{X}_i S_j(\hat{X}_j - X_j^{(k)})\,,\end{aligned}$$

(vi)
$$\begin{aligned}\mathcal{R}_i(\hat{X}_1, \hat{X}_2) = \;&(\mu I_n + A^T)(X_i^{(k+1)} - \hat{X}_i)\\ &+(\hat{X}_i - Y_i^{(k)})(\mu I - A + S_i Y_i^{(k)} + S_j Y_j^{(k)})\\ &+\hat{X}_i S_i(\hat{X}_i - Y_i^{(k)}) + \hat{X}_i S_j(\hat{X}_j - Y_j^{(k)})\,.\end{aligned}$$

Proof. The proof is completed by a direct calculation.

The following theorem is true. We accept it without a proof. The proof repeats similar proofs such as Theorem 2,[3].

Theorem 2 *Assume the matrix $-A$ is an M-matrix and $Q_i \geq 0, i = 1, 2$, and $S_i \leq 0\,, i = 1, 2\,, \mu < 0$, such that $(-\mu I - A)$ is an M-matrix and $\mu I - A$ is nonpositive. Assume there exist nonnegative matrices $\hat{X}_1, \hat{X}_2$, such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0\,, i = 1, 2$ and $-\mu I_n - A + S_1\hat{X}_1 + S_2\hat{X}_2$ is an M-matrix. The matrix sequences $\{X_1^{(k)}\,, X_2^{(k)}\}_{k=0}^{\infty}$ defined by (4)-(5) satisfy the following properties:*

(i) $\hat{X}_i, \geq X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ *for* $i = 1, 2\,, \quad k = 0, 1, \ldots;$

(ii)
$$\begin{aligned}&\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0\,, \quad \mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}) \leq 0\,,\\ &\mathcal{R}_i(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0\,, \quad i = 1, 2\,, \quad k = 0, 1, \ldots\,.\end{aligned}$$

(iii) *The matrix sequences $\{X_1^{(k)}\,, X_2^{(k)}\}_{k=0}^{\infty}$ converge to the nonnegative minimal solution $\tilde{X}_1, \tilde{X}_2$ to the set of Riccati equations $\mathcal{R}_1(X_1, X_2)) = 0\,, \mathcal{R}_2(X_1, X_2)) = 0$ with $\tilde{X}_i \leq \hat{X}_i$.*

(iv) *Moreover, if* $-A+S_1\hat{X}_1+S_2\hat{X}_2$ *and* $-D+\hat{X}_1S_1+\hat{X}_2S_2$ *are M-matrices, then the solution* $\tilde{\mathcal{X}} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ *is a left- right stabilizing solution of the nonsymmetric Nash Riccati equation* $\mathfrak{R}(\mathcal{X}) = 0$.

3 Numerical examples

We execute numerical experiments on several numerical examples in this section. We apply the above decoupled iterative methods to compute the stabilizing solution of the nonsymmetric Riccati equation (1), i.e. decoupled iteration ALIDI (2)-(3) and decoupled iteration (4)-(5).

From the computational point of view it is interesting to investigate the following decoupled iteration. We call it the Decoupled Iteration 2 (DI2):

$$Y_i^{(k)}(\mu I_n + A) = (\mu I - A^T + X_1^{(k)}S_1 + X_2^{(k)}S_2)X_i^{(k)} - Q_i\,, \tag{6}$$

$$(\mu I_n + A^T)X_i^{(k+1)} = Y_i^{(k)}(\mu I - A + S_1Y_1^{(k)} + S_2Y_2^{(k)}) - Q_i\,, \tag{7}$$

$i = 1, 2,\ X_1^{(0)} = X_2^{(0)} = 0\,,\quad k = 0, 1, 2, \dots\,,\ \mu < 0\,.$

The matrix coefficients in the above iteration are transposed to one another, i.e. $(\mu I_n + A)^T = \mu I_n + A^T$. In addition, $(\mu I_n + A^T)^{-1} = ((\mu I_n + A)^{-1})^T$. Note that the second recursive equation (5) is the same as (7). In order to compute $X_i^{(k+1)}, i = 1, 2, k = 0, 1, \dots$ one needs only one computation of a inverse matrix.

We have compared these three decoupled iterations. The matrix coefficients A, B_i, Q_i and R_{ii} for $i = 1, 2$ are defined using the Matlab description. All experiments were executed on Dell Computer with Processor Intel Core i7-1065G7 1.30 GHz. The numerical experiments are constructed following the approach applied in [4].

Example 1. The matrix coefficients are:
A=abs(randn(n))/9; s=max(abs(eig(A)))+3.5; μ is a parameter with $\mu < 0$;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1 = abs(randn(n,4))/2;
B_2 = 0.7*eye(n,n); B_2(n,n)=0.67;
Q_1=zeros(n,n); Q_1(1,1)=n/2; Q_1(n,n)=1.5;
for i=1:n-1, Q_1(i,i+1)=1/sqrt(n); Q_1(i+1,i)=1/sqrt(n); end
$Q_2 = 2 * Q_1$;
R_{11}=-10;
R_{22} = -eye(n,n); R_{22}(1,1)=-50; R_{22}(n,n)=-30;

We are executing this example for different values of n, and 100 runs are completed for each value of n. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}) = -Q_i \le 0\,,\quad i = 1, 2$, (i.e. the matrices are nonpositive).

Table 1 presents the computational results for different values of n.

Table - NEW VARIANT

	ALIDI (2)-(3)			DI1 (4)-(5)			DI2 (6)-(7)		
n	mIt	avIt	CPU	mIt	avIt	CPU	mIt	avIt	CPU
28	51	48.3	0.0075s	56	53	0.0071s	51	47.1	0.0064s
56	72	69.6	0.0622s	85	79.4	0.0555s	73	67.8	0.0445s

Example 2. The matrix coefficients are:
A=abs(randn(n))/10; s=max(abs(eig(A)))+1.5;

for i=1:n, A(i,i)=-(A(i,i))-s; end

B_1=abs(randn(n,1))/6;

B_2=eye(n,n); B_2(n,n)=n/5; B_2(1,1)=n/10; B_2(1,n)=randn/10;

Q_1=zeros(n,n); Q_1(1,1)=n/5; Q_1(n,n)=1/n;

Q_2=0.25 Q_1; R_{11}=-1.5;

R_{22}=-eye(n,n); R_{22}(1,1)=-57; R_{22}(n,n)=-27;

We change the dimension n: $n = 35, 60, 80, 100$. We have 100 runs for each value of n with each iterative formula. The results are described in Table 2.

Table 2. tol= 1e-12, $\gamma = -1.5$

	ALIDI (2)-(3)			DI1 (4)-(5)			DI2 (6)-(7)		
n	mIt	avIt	CPU	mIt	avIt	CPU	mIt	avIt	CPU
		$\gamma = -1.5$							
35	24	22.4	0.0086s	26	23.9	0.0045s	23	22.0	0.0039s
60	34	33.4	0.0337s	37	35.9	0.0279s	33	31.9	0.0227s
80	43	41.9	0.0747s	46	44.5	0.0619s	40	38.3	0.0527s
100	52	50.4	0.1686s	55	53.5	0.1423s	46	45.3	0.1126s

4 Conclusion

Decoupled iteration (6)-(7) is more effective than another iterations for the examples. However, we have no the convergence proof for the iteration (6)-(7).

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