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A Nash Equilibrium Strategy of Linear Quadratic Differential Games for Positive Systems with Markovian Jumping

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Abstract. This paper deals with the infinite horizon linear quadratic differential games for positive linear systems with Markovian jumping. Based on the concept of the stabilizing solution of a system of game theoretic algebraic Riccati type equations, a theorem on finding the optimal control strategies is derived. As the main result of this work, a set of conditions guaranteeing the existence of the stabilizing solution of the system of game theoretic algebraic Riccati equations is provided. A globally convergent iterative algorithm to solve the associated set of generalized algebraic Riccati equations is presented. The performances of the proposed algorithm are illustrated on some numerical examples.

Key Words: Linear quadratic games, Nash equilibrium, Positive systems with Markovian jumping, Set of game theoretic algebraic Riccati equations, Stabilizing solutions.

1 Introduction

Dynamic games represent an important research field of control theory with a high potential of applicability. Many situations in economics and management are characterized by multiple decision makers/players. The theory of dynamic games is one of the theories which conceptualizes these types of situations. Examples of dynamic games in economics and management science can be found e.g. in [1, 2, 3, 4, 5]. Recently, the theory of linear quadratic (LQ) differential games based on Riccati equations has received much attention. For the readers convenience we refer to [6, 7, 8, 9, 10].

In the case when the LQ differential game is defined on an infinite time horizon, each player has to take into account two tasks in the analysis and synthesis of the optimal strategy:

a) a common task: the adopted strategy has to preserve the asymptotically stable behavior of the trajectories of the dynamic system;

b) individual task: the adopted strategy must optimize his own performance criterion.

It is known that (see e.g. [9, 10]) the closed loop Nash equilibrium strategy is obtained in a linear state feedback form whose gain matrices are computed based on the solution of a system of algebraic Riccati equations known as system of game theoretic algebraic Riccati equations (SGTAREs). Having in mind the first task stated above, it is clear that the solution of SGTARE involved in the computation of the gain matrices of the Nash equilibrium strategy on an infinite time horizon, must be a stabilizing solution of the Riccati equation under consideration. A comprehensive theory regarding the problem of the existence and numerical computation of the stabilizing solution of a SGTARE occurring in connection with the closed loop LQ differential game with at least two players, is not yet available as it is the case of the well known algebraic Riccati equation arising in classical LQ regulator problem. In [9], one shows that even in a simple case of a one dimensional LQ differential game, the uniqueness of the stabilizing solution of the SGTARE is not guaranteed. This makes more difficult the development of a general methodology for solving the problem of the existence and the numerical calculus of the stabilizing solution. Some advances in solving these problems were obtained in the special case of positive systems (see e.g. [4, 7, 8, 15]).

In the present paper we consider a class of stochastic LQ differential games for positive linear systems with Markovian jumping. Usually, Markov Jump Linear Systems (MJLSs) are a popular class of stochastic systems that are well suited to describe dynamics characterized by random jumps between subsystems induced by external causes, such as random faults, unexpected events, uncontrolled configuration changes. For the readers convenience we refer to [11]. We consider the case in which all the subsystems belong to the class of linear positive systems whose state variables remain nonnegative whenever they start in the positive orthant. Positive systems and their applications occur naturally in ecological and economic systems [6, 7].

In this paper, the theory of linear quadratic differential games and properties of positive systems are employed to derive sufficient conditions for the existence of closed-loop Nash equilibrium strategy. As in the deterministic case considered in Section 7 and Section 8 from [12], we shall use the special properties of the positive systems to prove the existence of the stabilizing solution of the corresponding SGTARE and consequently, the existence of a closed-loop Nash equilibrium strategy. To derive a set of sufficient conditions which guarantee the existence of the stabilizing solution, we introduce the concept of strong stabilizing solution of SGTARE involved in computation of the gain matrices of a closed-loop Nash equilibrium strategy. This new type of solution of SGTARE is characterized based on properties of the spectrum of the Freche derivative of the Riccati operator. This fact allows us to construct a sequence of approximations whose limit is just the strong stabilizing solution of the SGTARE. Employing the positivity properties of the coefficients of the considered system we succeed to prove that the strong stabilizing solution of SGTARE is just a stabilizing solution of this kind of Riccati equations. The sequence of approximations involved in the proof of the existence of the strong stabilizing solution may be used for numerical computation of this solution.

The outline of the paper is as follows: Section 2 contains the model description and the statement of the concept of closed-loop Nash equilibrium. In Section 3 we introduce the concept of the stabilizing solution of the SGTARE and show the role of this solution in the derivation of a closed-loop Nash equilibrium strategy. Section 4 is devoted to the problem of the existence of the stabilizing solution of SGTARE. First, we introduce the concept of strong stabilizing solution of SGTARE and we show that under some suitable assumptions, this kind of solution is a stabilizing solution in the sense of definition from Section 3. Further, we provide a set of sufficient conditions which guarantee the existence of a strong stabilizing solution, and also a stabilizing solution. In the last section, we discuss some procedural issues regarding the numerical computation of the stabilizing solution of SGTARE under consideration. The performance of the iterative procedures are illustrated by numerical examples.

2 Problem formulation

Let us consider the linear quadratic (LQ) differential game described by the dynamic system:

$$\dot{x}(t) = A(\eta_t)\, x(t) + \sum_{k=1}^{N} B_k(\eta_t)\, u_k(t)\,, \quad x(0) = x_0\,, \tag{1}$$

and the set of performance criteria

$$J_j(u_1, \ldots, u_N; x_0) = E \int_0^\infty \left[x^T(t) M_j(\eta_t) x(t) + \sum_{\ell=1}^{N} u_\ell^T(t) R_{j\ell}(\eta_t) u_\ell(t) \right] dt \tag{2}$$

$1 \leq j \leq N$, where $x(t) \in \mathbb{R}^n$ is the state at time instance t and $u_k : [0, \infty) \to \mathbb{R}^{m_k}$ are the control inputs (sometimes named the policy or the strategy that are available for the player $\mathcal{P}_k, 1 \leq k \leq N$).

According with the terminology used in the theory of dynamical games, the player $\mathcal{P}_k$ is an agent or a decision maker which chose the strategy $u_k(t), t \geq 0$ from the class of admissible strategies in order to optimize his own performance criterion (utility function). There are two main types of admissible strategies: open-loop admissible strategies and closed-loop admissible strategies. For precise definitions of these types of strategies in the deterministic case, we refer to [9] or [10]

In (1) and (2) the sequence $\{\eta_t\}_{t\geq 0}$ is a standard right continuous Markov process defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in the finite set $\mathcal{D} = \{1, 2, \ldots, d\}$ and having the transition semigroup, $P(t) = e^{Qt}\,, t \geq 0$ where $Q \in \mathbb{R}^{d\times d}$ is a matrix whose entries (q_{ij}) satisfy the condition

$$q_{ij} \geq 0 \;\text{ if }\; i \neq j, \quad \sum_{j=1}^{d} q_{ij} = 0, \quad i \in \mathcal{D}\,, \tag{3}$$

(for more details see [13, 14]).

In this paper we shall write $A(i), B_k(i), M_k(i), R_{k\ell}(i)$ instead of $A(\eta_t), B(\eta_t), M_k(\eta)t)$, $R_{k\ell}(\eta_t)$ whenever the Markov process η_t lies in the state $i \in \mathcal{D}$. In this way, the dynamical system (1) may be viewed as a multi-model dynamical system described by d deterministic subsystems. The switching from a subsystem to another one is driven by the Markov process η_t. In the special case $d = 1$ the system (1) becomes a deterministic dynamical system.

Throughout the paper $\succcurlyeq$ stands for the component wise order relation. This means that if $z = (z_1, \ldots, z_p)^T \in \mathbb{R}^p$ then $z \succcurlyeq 0$ if and only if $z_\ell \geq 0, 1 \leq \ell \leq p$. Also, if $C \in \mathbb{R}^{p\times p}$ is a matrix, then $C \succcurlyeq 0$ if and only if all its elements c_{ij} are nonnegative real numbers.

Definition 1 *The system of linear differential equations (1) is named* **positive system** *if for any $x_0 \succcurlyeq 0$ and for any admissible strategy $u_k : [0, \infty) \to \mathbb{R}^{m_k}$ with the property that $u_k(t) \succcurlyeq 0$ for all $t \geq 0, 1 \leq k \leq N$, it follows that $x(t) \succcurlyeq 0, t \geq 0$.*

Control problems involving positive systems occur in a natural way in connection with some problems from economy and different other domains from the real life (see, for example [5, 6, 7, 15] for the deterministic case and [16, 17] for the stochastic case).

Remark 2 *A sufficient condition for the system (1) to be a positive system is that for each $i \in \mathcal{D}$, $A(i)$ is a Metzler matrix and $B_k(i) \succcurlyeq 0, 1 \leq k \leq N$. We recall that $A(i)$ is named Metzler matrix if its off diagonal entries are nonnegative (for details see e.g. [6, 17]).*

In this work we consider the so called closed-loop Nash strategies. This means that each player has access to the strategies in a linear state feedback form

$$u_k(t) = F_k(\eta_t)\, x(t)\,, \tag{4}$$

where $F_k(i) \in \mathbb{R}^{m_k\times n}\,, 1 \leq k \leq N\,, i \in \mathcal{D}$.

Since the dynamical system (1) is linear, the assumption that the closed-loop strategies (4) are in a linear state feedback form is a natural one in order to preserve the linear character of the corresponding closed loop system. In the special case $d = 1$, the admissible strategies of type (4) recover the strategies considered in [9] for a deterministic closed-loop LQ differential game.

Definition 3 *We say that a closed-loop strategy (4) stabilizes the dynamical system (1) in the mean square sense, if the resulting system (obtained when (4) is plugged in (1))*

$$\dot{x}(t) = (A(\eta_t) + \sum_{k=1}^{N} B_k(\eta_t)F_k(\eta_t))x(t) \tag{5}$$

is exponentially stable in mean square (ESMS). This means that there exist $\beta \geq 1, \alpha > 0$, such that the solutions of the differential equation (5) satisfy:

$$E[|x(t)|^2] \leq \beta e^{-\alpha t}|x_0|^2$$

for all $t \geq 0$ and all initial states $(x_0, \eta_0) \in \mathbb{R}^n \times \mathcal{D}$.

Here and after $|\cdot|$ stands for the Euclidian norm on $\mathbb{R}^n$, i.e. $|x| = (x^T x)^{1/2}$ for all $x \in \mathbb{R}^n$.

We denote $\mathbb{F}_k = (F_k(1), \ldots, F_k(d)), 1 \le k \le N$. Substituting $u_k(t)$ in (2) using (4) we obtain the following form of the performance criterion

$$\begin{aligned} &J_j(\mathbb{F}_1, \ldots, \mathbb{F}_N; x_0) = \\ &E \int_0^\infty x^T(t) \left(M_j(\eta_t) + \sum_{\ell=1}^N F_\ell^T(\eta_t) R_{j\ell}(\eta_t) F_\ell(\eta_t) \right) x(t) dt \end{aligned} \tag{6}$$

$x(t)$ being the solution of (5) such that $x(0) = x_0$.

Denote $\mathfrak{F}_N$ the set of N-tuples $(\mathbb{F}_1, \ldots, \mathbb{F}_N)$ with $\mathbb{F}_k = (F_k(1), \ldots, F_k(d)), F_k(i) \in \mathbb{R}^{m_k \times n}, 1 \le k \le N, i \in \mathcal{D}$ having the property that the corresponding control of type (4) stabilizes the system (1) in mean-square sense and additionally, for each $i \in \mathcal{D}$, $A(i) + \sum_{k=1}^N B_k(i) F_k(i)$ is a Metzler matrix. A N-tuple $(\mathbb{F}_1, \ldots, \mathbb{F}_N)$ from $\mathfrak{F}_N$ will be named admissible N-tuple of feedback gains and the corresponding control (4) will be named admissible strategy.

In the characterization of the admissible strategies from above, we required that $A(i) + \sum_{k=1}^N B_k(i) F_k(i)$ be a Matzler matrix in order to be sure that the corresponding closed-loop system of type (5) is a positive system.

Definition 4 *We say that* $(\tilde{\mathbb{F}}_1, \ldots, \tilde{\mathbb{F}}_N) \in \mathfrak{F}_N$ *is a feedback Nash equilibrium if for each* $1 \le k \le N$*, we have that*

$$J_k(\tilde{\mathbb{F}}_1, \ldots, \tilde{\mathbb{F}}_{k-1}, \mathbb{F}_k, \tilde{\mathbb{F}}_{k+1}, \ldots, \tilde{\mathbb{F}}_N; x_0) \le J_k(\tilde{\mathbb{F}}_1, \ldots, \tilde{\mathbb{F}}_N; x_0), \tag{7}$$

for all $x_0 \in \mathbb{R}^n$ *and each* $\mathbb{F}_k = (F_k(1), \ldots, F_k(d))$ *such that* $(\tilde{\mathbb{F}}_1, \ldots, \tilde{\mathbb{F}}_{k-1}, \mathbb{F}_k, \tilde{\mathbb{F}}_{k+1}, \ldots, \tilde{\mathbb{F}}_N)$ *lies in* $\mathfrak{F}_N$.

From the definition of Nash equilibrium strategy, it follows that the aim of each player is to maximize his own performance criterion (utility function).

In the next section we shall provide a set of sufficient conditions which guarantee the existence of a feedback Nash equilibrium and we give an explicit formula of the feedback gains $\tilde{F}_k(i)$.

3 Feedback Nash equilibrium strategy

Based on the coefficients of the system (1) and weights matrices from (2) we introduce the following system of game theoretic algebraic Riccati equations (SGTARE):

$$\begin{aligned} &[A(i) - \sum_{l=1, l\neq k}^N S_l(i) X_l(i)]^T X_k(i) + X_k(i)[A(i) - \sum_{l=1, l\neq k}^N S_l(i) X_l(i)] - \\ &X_k(i) S_k(i) X_k(i) + \sum_{l=1, l\neq k}^N X_l(i) S_{kl}(i) X_l(i) + \sum_{j=1}^d q_{ij} X_k(j) + M_k(i) = 0 \end{aligned} \tag{8}$$

$1 \le k \le N, i \in \mathcal{D}$ where we denoted

$$\begin{aligned} S_k(i) &= B_k(i) R_{kk}^{-1}(i) B_k^T(i), \\ S_{kl}(i) &= B_l(i) R_{ll}^{-1}(i) R_{kl}(i) R_{ll}^{-1}(i) B_l^T(i). \end{aligned} \tag{9}$$

To this end we need to suppose that the matrices $R_{ll}(i)$ are invertible.

Definition 5 *A solution* $\tilde{\mathbf{X}} = ((\tilde{X}_1(1),\ldots,\tilde{X}_1(d)),\ldots,(\tilde{X}_N(1),\ldots,\tilde{X}_N(d)))$ *is named a stabilizing solution of SGTARE if it satisfies (8) and additionally, the following closed-loop system*

$$\dot{x}(t) = [A(\eta_t) + \sum_{k=1}^{N} B_k(\eta_t)\,\tilde{F}_k(\eta_t)]\,x(t) \tag{10}$$

is exponentially stable in mean-square (ESMS), where

$$\tilde{F}_k(i) = -R_{kk}^{-1}(i)B_k^T(i)\,\tilde{X}_k(i)\,, \quad 1 \le k \le N\,,\ i \in \mathcal{D}\,. \tag{11}$$

The main result of this section is:

Theorem 6 *Assume:*

a) for each $i \in \mathcal{D}, A(i)$ *is a Metzler matrix,* $B_k(i) \succcurlyeq 0\,, 1 \le k \le N$ *;*

b) $R_{kk}(i) < 0$ *(negative definite) and* $R_{kk}^{-1}(i) \prec 0,\ 1 \le k \le N\,, i \in \mathcal{D}$*.*

c) The SGTARE (8) has a stabilizing solution $\tilde{\mathbf{X}}$ *with the property that* $\tilde{X}_k(i) \succcurlyeq 0\,, 1 \le k \le N\,, i \in \mathcal{D}$*.*

Let $\tilde{\mathbb{F}}_k = (\tilde{F}_k(1),\ldots,\tilde{F}_k(d))$*, and* $\tilde{F}_k(i)$ *being introduced by (11). Under the considered assumptions* $(\tilde{\mathbb{F}}_1,\ldots,\tilde{\mathbb{F}}_N)$ *lies in* $\mathfrak{F}_N$ *and provides a feedback Nash equilibrium for the LQ differential game described by the dynamic system (1) and the cost functionals (6). Moreover,*

$$J_k(\tilde{\mathbb{F}}_1,\ldots,\tilde{\mathbb{F}}_N; x_0) = x_0^T\, E\,[\tilde{X}_k(\eta_0)]\,x_0\,. \tag{12}$$

Proof: From the considered assumptions together with (11) is obvious that $\tilde{F}_k(i) \succcurlyeq 0\,, B_k(i)\tilde{F}_k(i) \succcurlyeq 0\,, 1 \le k \le N,\ i \in \mathcal{D}$. Hence, $A(i) + \sum_{k=1}^{N} B_k(i)\,\tilde{F}_k(i),\ i \in \mathcal{D}$ are Metzler matrices. Thus, the exponential stability in mean square of the closed-loop system (10) allows us to conclude that $(\tilde{\mathbb{F}}_1,\ldots,\tilde{\mathbb{F}}_N) \in \mathfrak{F}_N$.

It remains to show that (7) is fulfilled. Let $1 \le k \le N$ be fixed. For $l \ne k$ we substitute $u_l(t)$ by $u_l(t) = \tilde{F}_l(\eta_t)\,x(t)$ both in (1) and (2), obtaining

$$\dot{x}(t) = \mathbb{A}(\eta_t)\,x(t) + B_k(\eta_t)\,u_k(t)\,, \quad x(0) = x_0\,, \tag{13}$$

and

$$\mathbb{J}_k(u_k; x_0) = E\int_0^\infty [x^T(t)\mathbb{M}_k(\eta_t)x(t) + u_k^T(t)R_{kk}(\eta_t)u_k(t)]\,dt \tag{14}$$

where we denoted

$$\mathbb{A}_k(i) = A(i) + \textstyle\sum_{l=1,l\ne k}^{N} B_l(i)\,\tilde{F}_l(i)\,,$$
$$\mathbb{M}_k(i) = M_k(i) + \textstyle\sum_{l=1,l\ne k}^{N} \tilde{F}_l^T(i)\,R_{kl}(i)\tilde{F}_l(i)\,.$$

With these notations (8) may be rewritten as follows:

$$\mathbb{A}_k^T(i)\,X_k(i) + X_k(i)\,\mathbb{A}_k(i) - X_k(i)S_k(i)X_k(i) + \sum_{j=1}^{d} q_{ij}X_k(j) + \mathbb{M}_k(i) = 0 \tag{15}$$

and the closed-loop system (10) be rewritten as:

$$\dot{x}(t) = [\mathbb{A}_k(\eta_t) + B_k(\eta_t)\,\tilde{F}_k(\eta_t)]x(t)\,. \tag{16}$$

From (10) and (16) we deduce that if $\tilde{\mathbf{X}}$ is a stabilizing solution of the SGTARE (8), then $\tilde{\mathbb{X}}_k = (\tilde{X}_k(1), \ldots, \tilde{X}_k(d))$ is the stabilizing solution of SARE (15). Using the square completion technique, one obtains that

$$\begin{aligned} \mathbb{J}_k(u_k; x_0) &= x_0^T E\,[\tilde{X}_k(\eta_0)]x_0 \\ + E\int_0^\infty [u_k(t) - \tilde{F}(\eta_k)\,x(t)]^T R_{kk}(\eta_t)\,[u_k(t) - \tilde{F}_k(\eta_t)\,x(t)]\,dt \end{aligned} \tag{17}$$

for any $u_k \in L^2_\eta(\mathbb{R}_+, \mathbb{R}^{m_k})$, having the additional property that the corresponding solution of the problem with given initial value (14) satisfies

$$\lim_{t\to\infty} E[|x_k(t)|^2] = 0.$$

Particulary, (17) is true for the inputs of the form $u_k(t) = F_k(\eta_t)\,x_k(t)$ for arbitrary $F_k(i) \in \mathbb{R}^{m_k\times n}$ with the property that $(\tilde{\mathbb{F}}_1, \ldots, \tilde{\mathbb{F}}_{k-1}, \mathbb{F}_k, \tilde{\mathbb{F}}_{k+1}, \ldots, \tilde{\mathbb{F}}_N) \in \mathfrak{F}_N$, $\mathbb{F}_k = (F_k(1), \ldots, F_k(d))$. The conclusion of the theorem follows now from (17), thus the proof is complete. □

Remark 7 *If $R_{kk}(i)$ is a symmetric Metzler matrix which is negative definite, then $R^{-1}_{kk}(i) \prec 0$ (see [7]).*

Since the stabilizing solution of SGTARE (8) plays a central role in the computation of the gain matrices of a feedback Nash equilibrium, an important problem is to provide conditions which guarantee the existence of the stabilizing solution of this kind of coupled algebraic Riccati equation. This problem will be addressed in detail in the next section.

4 Conditions for the existence of the stabilizing solution of SGTARE

4.1 Some preliminary

Let $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ be the linear subspace of a symmetric $n \times n$ matrices and $\mathcal{S}_n^d = \mathcal{S}_n \otimes \mathcal{S}_n \otimes \ldots \otimes \mathcal{S}_n$. The elements of the space $\mathcal{S}_n^d$ are finite sequences $\mathbb{X} = (X(1), \ldots, X(d))$ of symmetric matrices. We set $\mathfrak{X} = \mathcal{S}_n^d \times \ldots \times \mathcal{S}_n^d$ (N factors). The elements $\mathbf{X}$ of the space $\mathfrak{X}$ can be represented either in the form $\mathbf{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_N)$, where $\mathbb{X}_k \in \mathcal{S}_n^d$, or in the form $\mathbf{X} = ((X_1(1), \ldots, X_1(d)), \ldots, (X_N(1), \ldots, X_N(d)))$. On $\mathfrak{X}$ we introduce the inner product

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \sum_{k=1}^{N}\sum_{j=1}^{d} Tr[X_k(j)Y_k(j)] \tag{18}$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}$, $Tr[\cdot]$ stands for the trace of a matrix. One sees that $\mathfrak{X}$ equips with the inner product (18) is a real Hilbert space. On the space $\mathfrak{X}$ we introduce the order relation " $\succcurlyeq$ " induced by the convex cone $\mathfrak{X}_+ = \{\mathbf{X} \in \mathfrak{X} | X_k(i) \succcurlyeq 0\,, 1 \le k \le N, i \in \mathcal{D}\}$. Here, $X_k(i) \succcurlyeq 0$ means that all entries of the matrix $X_k(i)$ are nonnegative real numbers. $\mathfrak{X}_+$ is a closed convex cone with non empty interior and the norm induced by the inner product (18) is a monotone norm with respect to this cone. The interior $Int\mathfrak{X}_+$ consists of all elements $\mathbf{X} \in \mathfrak{X}_+$ with the property that $X_k(i) \succ 0, 1 \le k \le N, i \in \mathcal{D}$, where $X_k(i) \succ 0$

means that all entries of the matrix $X_k(i)$ are positive real numbers. We shall write $\mathbf{X} \succcurlyeq \mathbf{Y}$ if and only if $\mathbf{X} - \mathbf{Y} \in \mathfrak{X}_+$ and $\mathbf{X} \succ \mathbf{Y}$ if and only if $\mathbf{X} - \mathbf{Y} \in Int\mathfrak{X}_+$. Let us consider the linear operator $\mathbb{L} : \mathfrak{X} \to \mathfrak{X}$, $\mathbb{L}[\mathbf{X}] = (\mathbb{L}_1[\mathbf{X}], \dots, \mathbb{L}_N[\mathbf{X}])$, $\mathbb{L}_k[\mathbf{X}] = (\mathbb{L}_k[\mathbf{X}](1), \dots, \mathbb{L}_k[\mathbf{X}](d))$, with

$$\begin{aligned} \mathbb{L}_k[\mathbf{X}](i) \;&=\; \Gamma^T(i)X_k(i) + X_k(i)\Gamma(i) \\ &+ \textstyle\sum_{l=1,l\neq k}^{N} \left(\Xi_{lk}^T(i)\, X_l(i) + X_l(i)\, \Xi_{kl}(i)\right) + \sum_{j=1}^{d} q_{ij} X_k(j)\,, \end{aligned} \tag{19}$$

for all $\mathbf{X} = ((X_1(1), \dots, X_1(d)), \dots, (X_N(1), \dots, X_N(d)))$, where $\Gamma(i) \in \mathbb{R}^{n\times n}$ and $\Xi_{kl}(i) \in \mathbb{R}^{n\times n}$ are given matrices and the scalars q_{ij} satisfy (3).
Several properties of the operator $\mathbb{L}$ defined above are summarized in the next proposition:

Proposition 8 *Assume*
a) for each $i \in \mathcal{D}, \Gamma(i)$ is a Metzler matrix;
b) for each $1 \le k \le N, 1 \le l \le N, k \neq l, i \in \mathcal{D}, \Xi_{kl}(i)$ are positive matrices.
Under these conditions the following hold:

(i) the linear operator $\mathbb{L}$ defines a positive evolution on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$, that is , $e^{\mathbb{L}t}\, \mathfrak{X}_+ \subset \mathfrak{X}_+, t \ge 0$,

(ii) if the eigenvalues of the linear operator $\mathbb{L}$ are in the half plane $\mathbb{C}_- = \{z \in \mathbb{C} | Re\, z < 0\}$, then the system of stochastic linear differential equations $\dot{x}(t) = \Gamma(\eta_t)\, x(t)$ is ESMS, η_t being any right continuous standard Markov process having the states in the set $\mathcal{D} = \{1, 2, \dots, d\}$ and the transition semigroup $P(t) = e^{Qt}, t \ge 0$.

Proof: (i) Let consider the linear operators $\mathfrak{L}, \Theta : \mathfrak{X} \to \mathfrak{X}$ defined as follows: $\mathfrak{L}[\mathbf{X}] = (\mathfrak{L}_1[\mathbf{X}], \dots, \mathfrak{L}_N[\mathbf{X}]), \mathfrak{L}_k[\mathbf{X}] = (\mathfrak{L}_k[\mathbf{X}](1), \dots, \mathfrak{L}_k[\mathbf{X}](d))$, $\Theta[\mathbf{X}] = (\Theta_1[\mathbf{X}], \dots, \Theta_N[\mathbf{X}]), \Theta_k[\mathbf{X}] = (\Theta_k[\mathbf{X}](1), \dots, \Theta_k[\mathbf{X}](d))$ where

$$\mathfrak{L}_k[\mathbf{X}](i) = \Gamma^T(i)X_k(i) + X_k(i)\Gamma(i) + \sum_{j=1}^{d} q_{ij}\, X_k(j), \tag{20}$$

$$\Theta_k[\mathbf{X}](i) = \sum_{l=1,l\neq k}^{N} \Xi_{kl}^T\, X_l(i) + X_l(i)\, \Xi_{kl}(i), \tag{21}$$

for each $\mathbf{X} = ((X_1(1), \dots, X_1(d)), \dots, (X_N(1), \dots, X_N(d)))$. From (20) - (21) it follows that $\mathbb{L}[\mathbf{X}] = \mathfrak{L}[\mathbf{X}] + \Theta[\mathbf{X}]$, for $\mathbf{X} \in \mathfrak{X}$. From (21) and assumption b) in the statement, it follows immediately that Θ is a positive operator on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$, i.e. $\Theta\, \mathfrak{X}_+ \subset \mathfrak{X}_+$. Further, we show that $\mathfrak{L}$ introduced by (20) defines a positive evolution on $(\mathfrak{X}, \mathfrak{X}_+)$. To this end, we have to show that $e^{\mathfrak{L}\, t}\, \mathfrak{X}_+ \subset \mathfrak{X}_+$ for all $t \ge 0$. With other words we have to check that the solutions of the linear differential equation

$$\dot{\mathbf{X}}(t) = \mathfrak{L}[\mathbf{X}](t) \tag{22}$$

satisfies $\mathbf{X}(t) \succcurlyeq 0$ for all $t \ge 0$ if $\mathbf{X}(0) \succcurlyeq 0$.
The k-th component of the differential equation (22) is the following differential equation on the space $\mathcal{S}_n^d$:

$$\dot{\mathbb{X}}_k(t) = \mathfrak{L}_k[\mathbb{X}_k(t)] \tag{23}$$

According to (20) it follows that for any $1 \leq k \leq N$ we have:

$$\mathfrak{L}_k[\mathbb{X}_k] = \mathfrak{L}_\Gamma[\mathbb{X}_k]$$

for $\mathbb{X}_k \in \mathcal{S}_n^d$, where

$$\mathfrak{L}_\Gamma[\mathbb{X}_k](i) = \Gamma^T(i)X_k(i) + X_k(i)\Gamma(i) + \sum_{j=1}^{d} q_{ij}\, X_k(j), \tag{24}$$

for all $1 \leq i \leq d\,, \mathbb{X}_k = (X_k(1), \ldots, X_k(N))\,.$ Hence the differential equation (23) becomes:

$$\dot{\mathbb{X}}_k(t) = \mathfrak{L}_\Gamma[\mathbb{X}_k(t)]\,. \tag{25}$$

We show that the solutions of the differential equation (25) satisfy $\mathbb{X}_k(t) \succcurlyeq 0,\; t \geq 0$ if $\mathbb{X}_k(0) \succcurlyeq 0\,.$
We rewrite the linear operator $\mathfrak{L}_\Gamma$:

$$\mathfrak{L}_\Gamma[\mathbb{X}] = \mathfrak{L}_0[\mathbb{X}] + \mathfrak{L}_1[\mathbb{X}]\,,$$

where

$$\mathfrak{L}_0[\mathbb{X}](i) = \hat{\Gamma}^T(i)X(i) + X(i)\,\hat{\Gamma}(i) \tag{26}$$

and

$$\mathfrak{L}_1[\mathbb{X}](i) = \sum_{j=1, j\neq i}^{d} q_{ij}\, X(j) \quad\; 1 \leq i \leq d\,, \tag{27}$$

and $\mathbb{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_n^d$, with $\hat{\Gamma}(i) = \Gamma(i) + \frac{1}{2} q_{ii}\, I_n\,.$ It is obvious that $\hat{\Gamma}(i)$ is a Metzler matrix if $\Gamma(i)$ is a Metzler matrix. Based on the Theorem 2.5 in [6] one obtains that the entries of the matrices $e^{\hat{\Gamma}(i)\, t}$ are nonnegative real numbers for all $t \geq 0, 1 \leq i \leq d$. By direct calculations one obtains that

$$e^{\mathfrak{L}_0\, t}\mathbb{X} \;= \left(e^{\hat{\Gamma}^T(1)\, t} X(1)\, e^{\hat{\Gamma}(1)\, t}, \ldots, e^{\hat{\Gamma}^T(d)\, t} X(d)\, e^{\hat{\Gamma}(d)\, t}\right)$$

for all $t \geq 0, \mathbb{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_n^d$. Therefore, $e^{\mathfrak{L}_0\, t}\mathbb{X} \succcurlyeq 0, t \geq 0$ if $\mathbb{X} = (X(1), \ldots, X(d))$ $\in \mathcal{S}_n^d$ is such that $X(j) \succcurlyeq 0, 1 \leq j \leq d$. On the other hand, (3) and (27) yield $\mathfrak{L}_1[\mathbb{X}](i) \succcurlyeq 0, 1 \leq i \leq d$ if $\mathbb{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_n^d$ with $X(j) \succcurlyeq 0, 1 \leq j \leq d$. Applying Corollary 2.2.6 from [18], (see also Corollary 2.7 from [19]) in the case of sum $\mathfrak{L}_\Gamma = \mathfrak{L}_0 + \mathfrak{L}_1$ we deduce that

$$e^{\mathfrak{L}_\Gamma\, t}\mathbb{X}(i) \succcurlyeq 0\,, \tag{28}$$

for all $t \geq 0, 1 \leq i \leq d$ if $\mathbb{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_n^d$ is such that $X(j) \succcurlyeq 0, 1 \leq j \leq d$.
Further on, employing (22) - (25) we may infer that,

$$e^{\mathfrak{L}\, t}[\mathbf{X}] = \left(e^{\mathfrak{L}_\Gamma\, t}[\mathbb{X}_1], \ldots, e^{\mathfrak{L}_\Gamma\, t}[\mathbb{X}_N]\right)\,, \tag{29}$$

for all $t \geq 0$ and all $\mathbf{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_N) \in \mathfrak{X}$. From (27) - (28) we deduce that $e^{\mathfrak{L}\, t}\mathfrak{X}_+ \subset \mathfrak{X}_+$, $t \geq 0$. Invoking again Corollary 2.2.6 from [18] or Corollary 2.7 from [19] in the case of the sum $\mathbb{L} = \mathfrak{L} + \Theta$ we may conclude that the operator $\mathbb{L}$ generates a positive evolution on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$. Thus, we have shown that assertion (i) in the statement holds.

To prove (ii), let us remark that $(\mathbb{L} - \mathfrak{L})\, \mathfrak{X}_+ \subset \mathfrak{X}_+$, which means that $\mathbb{L} \succcurlyeq \mathfrak{L}$. Applying Proposition 2.3.2 (ii) together with Corollary 2.3.9 from [18], we deduce that the eigenvalues of the operator $\mathfrak{L}$ are placed in the half plane $\mathbb{C}_-$ if the eigenvalues of the operator $\mathbb{L}$ are in the half plane $\mathbb{C}_-$. Since the eigenvalues of the operator $\mathfrak{L}_\Gamma$ are through the eigenvalues of the operator $\mathfrak{L}$, we conclude that the solutions of the differential equation (25) are exponentially stable. Bearing in mind that $\mathfrak{L}_\Gamma$ is the Lypunov type operator, associated to the system of stochastic linear differential equations

$$\dot{x}(t) = \Gamma(\eta_t)\, x(t) \tag{30}$$

we may conclude that the zero solution of (30) is ESMS. Thus the proof is complete. □

4.2 The strong stabilizing solution and the stabilizing solution to SGTARE

It is known that even in some simpler case, as the deterministic scalar case (see for example [9]), the uniqueness of the stabilizing solution of a SGTARE involved on closed-loop Nash equilibrium is not guaranteed. That is why, the problem of derivation of sufficient conditions for the existence of the stabilizing solution for SGTAREs of type (8) is a difficult problem.

In this section, we introduce the concept of strong stabilizing solution of (8) and we show that under some assumptions the strong stabilizing solution is a stabilizing solution of (8), too. The main result proved in this section will provide a set of conditions which guarantee the existence of a strong stabilizing solution of SGTARE (8). In this way, we obtain in an indirect manner, a set of conditions which guarantee the existence of the stabilizing solution of (8).

In order to facilitate the definition of the concept of the strong stabilizing solution, let us rewrite (8) in a compact form, as follows:

$$\mathfrak{R}[\mathbf{X}] \;=\; 0\,, \tag{31}$$

where $\mathfrak{R}[\mathbf{X}] = (\,\mathfrak{R}_1[\mathbf{X}], \ldots, \mathfrak{R}_N[\mathbf{X}]\,)$ with $\mathbf{X} \;\to\; \mathfrak{R}_k[\mathbf{X}] \;:\; \mathfrak{X} \to \mathcal{S}_n^d$, and $\mathfrak{R}_k[\mathbf{X}] = (\,\mathfrak{R}_k[\mathbf{X}](1), \ldots, \mathfrak{R}_k[\mathbf{X}](d))$, $\mathfrak{R}_k[\mathbf{X}](i)$ being described by the left hand side of (8). By direct calculation, one obtains the following compact form of the left hand side of (8) and consequently of $\mathfrak{R}_k[\mathbf{X}](i)$:

$$\begin{aligned} \mathfrak{R}_k[\mathbf{X}](i) \;&=\; A^T(i)\, X_k(i) + X_k(i)A(i) + \textstyle\sum_{j=1}^d q_{ij} X_k(j) \\ &\quad + \mathbb{S}_i^T(\mathbf{X})\Pi^k(i)\mathbb{S}_i(\mathbf{X}) + M_k(i)\,, \end{aligned} \tag{32}$$

$1 \le k \le N, i \in \mathcal{D}$, where $\mathbf{X} \to \mathbb{S}_i(\mathbf{X}) : \mathfrak{X} \to \mathbb{R}^{nN \times n}$ is defined by

$$\mathbb{S}_i(\mathbf{X}) = (X_1(i) \;\; X_2(i) \;\; \ldots \;\; X_N(i))^T \tag{33}$$

$\mathbf{X} = ((X_1(1), \ldots, X_1(d)), \ldots, (X_N(1), \ldots, X_N(d))) \in \mathfrak{X}$. In (32) $\Pi^k(i) \in \mathbb{R}^{nN \times nN}$ has the block structure

$$\Pi^k(i) = \begin{pmatrix} \Pi^k_{11}(i) & \Pi^k_{12}(i) & \dots \Pi^k_{1N}(i) & \\ {\Pi^k_{12}}^T(i) & \Pi^k_{22}(i) & \dots \Pi^k_{2N}(i) & \\ \dots & \dots & \dots & \dots \\ {\Pi^k_{1N}}^T(i) & {\Pi^k_{2N}}^T(i) & \dots \Pi^k_{NN}(i) & \end{pmatrix} \tag{34}$$

where $\Pi^k_{lj}(i) \in \mathcal{S}_n$ are defined as follows:

$$\begin{array}{l} \Pi^k_{ll}(i) = S_{kl}(i), \quad \text{if} \;\; l \in \{1,2,\dots,N\} \setminus \{k\} \\ \Pi^k_{lk}(i) = -S_l(i), \quad 1 \le l \le N \\ \Pi^k_{kl}(i) = -S_l(i), \quad k+1 \le l \le N \\ \Pi^k_{l_1 l_2}(i) = 0, \quad \text{if} \;\; (l_1,l_2) \notin \{(l,l),(l,k),(k,l) : 1 \le l \le N\} \end{array} \tag{35}$$

$S_{kl}(i), S_l(i)$ being introduced via (9).

Remark 9 *a) In the special case* $N = 2, d \ge 2$, *we have* $\mathbf{X} = ((X_1(1),\dots,X_1(d)),(X_2(1),\dots,X_2(d))) \in \mathfrak{X} = \mathcal{S}^d_n \times \mathcal{S}^d_n$ *and*

$$\mathbb{S}_i(\mathbf{X}) = \begin{pmatrix} X_1(i) \\ X_2(i) \end{pmatrix}, \Pi^1(i) = \begin{pmatrix} -S_1(i) & -S_2(i) \\ -S_2(i) & S_{12}(i) \end{pmatrix}, \;\; \Pi^2(i) = \begin{pmatrix} S_{21}(i) & -S_1(i) \\ -S_1(i) & -S_2(i) \end{pmatrix}.$$

b) From (35) become clear that $\Pi^k(i) \succcurlyeq 0$ *if* $R^{-1}_{kk}(i) \prec 0, R_{kl}(i) \succcurlyeq 0$ *and* $B_k(i) \succcurlyeq 0$, $1 \le k \le N, 1 \le l \le N, k \ne l, i \in \mathcal{D}$.

From (32) one sees that the operator valued function $\mathfrak{R}[\cdot]$ is Frechet differentiable on the whole space $\mathfrak{X}$. By direct calculation, based on (32) , we obtain that the Frechet derivative $\mathfrak{R}'[\mathbf{X}]$ is described by

$$\begin{array}{rcl} (\mathfrak{R}'_k[\mathbf{X}]\mathbf{Y})\,(i) & = & A^T(i)Y_k(i) + Y_k(i)A(i) + \sum_{j=1}^d q_{ij}Y_k(j) \\ & & +\mathbb{S}^T_i(\mathbf{Y})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}) + \mathbb{S}^T_i(\mathbf{X})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{Y}) \end{array} \tag{36}$$

for all $\mathbf{X} = ((X_1(1),\dots,X_1(d)),\dots,(X_N(1),\dots,X_N(d))) \in \mathfrak{X}$ and $\mathbf{Y} = ((Y_1(1),\dots,Y_1(d)),\dots,(Y_N(1),\dots,Y_N(d))) \in \mathfrak{X}$. Further on, (33)-(34) allow us to rewrite (36) in the form

$$\begin{array}{l} (\mathfrak{R}'_k[\mathbf{X}]\mathbf{Y})(i) = (A(i) - \sum_{l=1}^N S_l(i)X_l(i))^T Y_k(i) + Y_k(i)(A(i) - \sum_{l=1}^N S_l(i)X_l(i)) \\ + \sum_{l=1, l\ne k}^N [(X_l(i)S_{kl}(i) - X_k(i)S_l(i))Y_l(i) + Y_l(i)(S_{kl}(i)X_l(i) - S_l(i)X_k(i))] \\ + \sum_{j=1}^d q_{ij}Y_k(j) \end{array} \tag{37}$$

Remark 10 *Taking*

$$\begin{aligned}\Gamma(i) &= A(i) - \textstyle\sum_{l=1}^{N} S_l(i)\, X_l(i) \\ \Xi_{kl}(i) &= S_{kl}(i) X_l(i) - S_l(i) X_k(i)\,,\end{aligned} \tag{38}$$

we obtain that the derivative $\mathfrak{R}'[\mathbf{X}] : \mathfrak{X} \to \mathfrak{X}$ *is a linear operator of type (19).*

Now, we are in position to introduce the concept of strong stabilizing solution of SGTARE (8).

Definition 11 *We say that a solution* $\tilde{\mathbf{X}} = ((\tilde{X}_1(1), \dots, \tilde{X}_1(d)), \dots, (\tilde{X}_N(1), \dots, \tilde{X}_N(d)))$ *of (8) is named* **strong stabilizing solution** *if the eigenvalues of the linear operator* $\mathfrak{R}'[\tilde{\mathbf{X}}]$ *are located in the half plane* $\mathbb{C}_-$.

The next result provides a set of conditions which guarantee that a strong stabilizing solution of (8) is a stabilizing solution in the sense of Definition 5.

Proposition 12 *Assume:*

a) for each $i \in \mathcal{D}, A(i)$ *is a Metzler matrix and* $B_k(i) \succcurlyeq 0,\ 1 \le k \le N$;

b) $R_{kl}(i) = R_{kl}^T(i) \succcurlyeq 0$, *if* $l \ne k$ *and* $R_{ll}(i) = R_{ll}^T(i)$ *are invertible and* $R_{ll}^{-1}(i) \prec 0, 1 \le k, l \le N, i \in \mathcal{D}$;

c) the SGTARE (8) has s strong stabilizing solution $\tilde{\mathbf{X}} \succcurlyeq 0$.

Under these conditions $\tilde{\mathbf{X}}$ *is a stabilizing solution of (8) in the sense of Definition 5.*

Proof: Based on (38) we may infer that under the assumptions in the statement, the linear operator $\mathfrak{R}'[\mathbf{X}] : \mathfrak{X} \to \mathfrak{X}$ satisfies the assumptions of Proposition 8. Thus, applying Proposition 8 (ii) in the case of the linear operator $\mathbb{L} = \mathfrak{R}'[\tilde{\mathbf{X}}]$ we deduce that if the eigenvalues of the linear operator $\mathfrak{R}'[\tilde{\mathbf{X}}]$ are placed in the half plane $\mathbb{C}_-$ then the zero solution of the stochastic linear differential equation

$$\dot{x}(t) = [A(\eta_t) - \textstyle\sum_{l=1}^{N} S_l(\eta_t)\, \tilde{X}_l(\eta_t)]\, x(t) \tag{39}$$

is ESMS. Based on (11) we deduce that the system (39) coincides with the system of type (10), associated to the solution $\tilde{\mathbf{X}}$. This allows us to conclude that $\tilde{\mathbf{X}}$ is a stabilizing solution of (8). Thus the proof is complete. □

Let us recall an auxilliary result, which will be repeatedly used in the proof of the main result of this section.

Lemma 13 *For arbitrary* $\mathbf{X} = ((X_1(1), \dots, X_1(d)), \dots, (X_N(1), \dots, X_N(d)))$ *and* $\mathbf{W} = ((W_1(1), \dots, W_1(d)), \dots, (W_N(1), \dots, W_N(d))) \in \mathfrak{X}$, *the following hold:*

$$\begin{aligned} a)\ \mathfrak{R}_k[\mathbf{X}](i) &= A^T(i)X(i) + X(i)A(i) + \textstyle\sum_{j=1}^{d} q_{ij} X(j) \\ &\quad + \mathbb{S}_i^T(\mathbf{W})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{X}) + \mathbb{S}_i^T(\mathbf{X})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{W}) \\ &\quad + \mathbb{S}_i^T(\mathbf{X}-\mathbf{W})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{X}-\mathbf{W}) - \mathbb{S}_i^T(\mathbf{W})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{W}) + M_k(i) \\ &= (\mathfrak{R}'_k[\mathbf{W}]\mathbf{X})\,(i) + \mathbb{S}_i^T(\mathbf{X}-\mathbf{W})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{X}-\mathbf{W}) \\ &\quad - \mathbb{S}_i^T(\mathbf{W})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{W}) + M_k(i), \quad i \in \mathcal{D},\ 1 \le k \le N\,. \end{aligned}$$

$$b)\ (\mathfrak{R}'_k[\mathbf{W}]\mathbf{X})(i) = (\mathfrak{R}'_k[\mathbf{Y}]\mathbf{X})(i) + \mathbb{S}_i^T(\mathbf{W}-\mathbf{Y})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}) + \mathbb{S}_i^T(\mathbf{X})\,\Pi^k(i)\,\mathbb{S}_i^T(\mathbf{W}-\mathbf{Y})$$

$i \in \mathcal{D},\ 1 \leq k \leq N, \mathbf{Y} = ((Y_1(1),\dots,Y_1(d)),\dots,(Y_N(1),\dots,Y_N(d))) \in \mathfrak{X}$.

Proof: a) follows immediately from (32)-(36) and b) follows from (36). □

Now, we introduce two subsets of $\mathfrak{X}$ which will be involved in the statement and the proof of the main result of this section. Let

$$\mathfrak{H} = \{\mathbf{H} \in \mathfrak{X}\ |\mathbf{H} \succcurlyeq 0,\ \mathfrak{R}_k[\mathbf{H}](i) \preccurlyeq 0,\ \text{for all } 1 \leq k \leq N,\ i \in \mathcal{D}\} \tag{40}$$

$$\tilde{\mathfrak{H}} = \{\mathbf{H} \in \mathfrak{X}\ |\mathbf{H} \succcurlyeq 0,\ \mathfrak{R}_k[\mathbf{H}](i) \prec 0,\ \text{for all } 1 \leq k \leq N,\ i \in \mathcal{D}\} \tag{41}$$

Remark 14 *a) $\tilde{\mathfrak{H}} \subset \mathfrak{H}$; b) the set $\mathfrak{H}$ contains all solutions $\mathbf{X} = ((X_1(1),\dots,X_1(d)),\dots,(X_N(1),\dots,X_N(d)))$ of (8) satisfying $X_k(i) \succcurlyeq 0$ for all $1 \leq k \leq N,\ i \in \mathcal{D}$.*

The main result of this section is:

Theorem 15 *Assume: a) for each $i \in \mathcal{D}, A(i)$ is a Metzler matrix and the system of linear differential equations*

$$\dot{x}(t) = A(\eta_t)\,x(t) \tag{42}$$

is ESMS.

b) $B_l(i) \succcurlyeq 0,\ \ R_{kl}(i) = R_{kl}^T(i) \succcurlyeq 0,\ \ M_l(i) = M_l^T(i) \succcurlyeq 0,\ \ 1 \leq l \leq N,\ 1 \leq k \leq N,\ k \neq l,\ i \in \mathcal{D}$;

c) $R_{ll}(i) = R_{ll}^T(i) < 0$ (negative definite) and $R_{ll}^{-1}(i) \prec 0$ (componentwise), $1 \leq l \leq N, i \in \mathcal{D}$;

d) the set $\tilde{\mathfrak{H}}$ is not empty.

Under these conditions, the SGTARE (8) has a strong stabilizing solution $\tilde{\mathbf{X}} = ((\tilde{X}_1(1),\dots,\tilde{X}_1(d)),\dots,(\tilde{X}_N(1),\dots,\tilde{X}_N(d)))$ such that $\tilde{X}_k(i) \succcurlyeq 0, 1 \leq k \leq N, i \in \mathcal{D}$. Moreover, if $\mathbf{Y} = ((Y_1(1),\dots,Y_1(d)),\dots,(Y_N(1),\dots,Y_N(d)))$ is another solution of (8) with $Y_k(i) \succcurlyeq 0$, then we have $Y_k(i) \succcurlyeq \tilde{X}_k(i) \succcurlyeq 0, 1 \leq k \leq N, i \in \mathcal{D}$.

Proof: The strong stabilizing solution $\tilde{\mathbf{X}}$ is obtained as a limit of a sequence of iterations $\{\mathbf{X}^{(p)}\}_{p\geq 1}$. To obtain the iterations $\mathbf{X}^{(p)}$ one solves the linear equation on $\mathfrak{X}$:

$$\mathfrak{R}'[\mathbf{X}^{(p-1)}](\mathbf{X}^{(p)} - \mathbf{X}^{(p-1)}) + \mathfrak{R}[\mathbf{X}^{(p-1)}] = 0$$

which is just the Newton method specialized to the case of equation (31). Employing (37) and Lemma 13 (i) one obtains the following component wise version of the previous equation:

$$\mathfrak{R}'[\mathbf{X}^{(p-1)}]\,\mathbf{X}^{(p)} + \mathbf{\Phi}^{(p-1)} = 0 \tag{43}$$

where $\mathbf{\Phi}^{(p-1)} = ((\Phi_1^{(p-1)}(1),\dots,\Phi_1^{(p-1)}(d)),\dots,(\Phi_N^{(p-1)}(1),\dots,\Phi_N^{(p-1)}(d)))$ with

$$\Phi_k^{(p-1)}(i) = M_k(i) - \mathbb{S}_i^T(\mathbf{X}^{(p-1)})\Pi^k(i)\mathbb{S}_i(\mathbf{X}^{(p-1)}), 1 \leq k \leq N, i \in \mathcal{D}. \tag{44}$$

The proof of the convergence of the sequence of iterations is divided in four main steps.

Step 1. Let $\mathbf{X}^{(1)} = ((X_1^{(1)}(1), \ldots, X_1^{(1)}(d)), \ldots, (X_N^{(1)}(1), \ldots, X_N^{(1)}(d)) \in \mathfrak{X}$ be the solution of the equation

$$\begin{aligned} & A(i)^T X_k^{(1)}(i) + X_k^{(1)}(i)\, A(i) + \textstyle\sum_{j=1}^d q_{ij} X_k^{(1)}(j) + M_k(i) = 0\,, \\ & i \in \mathcal{D},\ 1 \le k \le N\,. \end{aligned} \tag{45}$$

Let us remark that (45) consists of N uncoupled equations of the form

$$\mathcal{T}[\mathbb{X}_k^{(1)}] + \mathbb{M}_k = 0 \tag{46}$$

where $\mathcal{T} : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is the linear operator of Lyapunov type described by $\mathcal{T}[\mathbb{X}](i) = A(i)^T X(i) + X(i)\, A(i) + \sum_{j=1}^d q_{ij} X(j)\,,\ 1 \le i \le d$ for all $\mathbb{X} = (X(1), \ldots, X(d)) \in \mathcal{S}_n^d$.

From Theorem 3.25 from [11] we infer that the system (42) is ESMS if and only if the eigenvalues of the operator $\mathcal{T}$ are in the left half plane $\mathbb{C}_-$. Since the operator $\mathcal{T}$ defined above coincides with (24) when $\Gamma(i) = A(i)$ we may obtain as in the proof of Proposition 8 that the operator $\mathcal{T}$ defines a positive evolution on the space $\mathcal{S}_n^d$ equiped with the componentwise ordered relation. So, applying Theorem 2.3.7 (iii), (iv) [18] in the special case of equation (46) we may deduce that under the considered assumptions this equation has an unique solution $\mathbb{X}_k^{(1)} = (X_k^{(1)}(1), \ldots, X_k^{(1)}(d)) \in \mathcal{S}_n^d$ with the property that $X_k^{(1)}(i) \succcurlyeq 0\,, 1 \le i \le N$. Hence, $\mathbf{X}^{(1)}$ can be obtained taking $\mathbf{X}^{(1)} = (\mathbb{X}_1^{(1)}, \ldots, \mathbb{X}_N^{(1)})$.

Further on, we show that if $\mathbf{H} \in \mathfrak{H}$ we have

$$X_k^{(1)}(i) \preccurlyeq H_k(i) \quad 1 \le k \le N\,, \quad i \in \mathcal{D}\,. \tag{47}$$

For each $1 \le k \le N$ and $i \in \mathcal{D}$, we set $\hat{M}_k(i) =: -\mathfrak{R}[\mathbf{H}](i)$. From (40) and (41) respectively, one sees that $\hat{M}_k(i) \succcurlyeq 0$ if $\mathbf{H} \in \mathfrak{H}$ and $\hat{M}_k(i) \succ 0$ if $\mathbf{H} \in \tilde{\mathfrak{H}}$. So, we obtained that $\mathbf{H}$ solves the following equation

$$\mathfrak{R}[\mathbf{H}] + \hat{\mathbf{M}} = 0 \tag{48}$$

where $\hat{\mathbf{M}} = ((\hat{M}_1(1), \ldots, \hat{M}_1(d)), \ldots, (\hat{M}_N(1), \ldots, \hat{M}_N(d)))$. Employing (32) and (48) we obtain that $\mathbf{H}$ satisfies the equation:

$$\begin{aligned} & A(i)^T H_k(i) + H_k(i)\, A(i) + \textstyle\sum_{j=1}^d q_{ij} H_k(j) + \\ & \mathbb{S}_i^T(\mathbf{H})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{H}) + M_k(i) + \hat{M}_k(i) = 0, \quad 1 \le k \le N\,, \quad i \in \mathcal{D}\,. \end{aligned} \tag{49}$$

Subtracting (45) from (49) we obtain that $\mathbf{H} - \mathbf{X}^{(1)}$ satisfies the equation:

$$\begin{aligned} & A(i)^T (H_k(i) - X_k^{(1)}(i)) + (H_k(i) - X_k^{(1)}(i))\, A(i) \\ & + \textstyle\sum_{j=1}^d q_{ij} (H_k(j) - X_k^{(1)}(j)) + K_k(i) = 0\,, \end{aligned} \tag{50}$$

where $K_k(i) = \mathbb{S}_i^T(\mathbf{H})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{H}) + \hat{M}_k(i)$.

According to Remark 4.1(b) and the definition of the set $\mathfrak{H}$ given in (40) we may deduce that $K_k(i) \succcurlyeq \hat{M}_k(i) \succcurlyeq 0, i \in \mathcal{D}$. Using again the fact that $\mathcal{T}$ is a linear operator which defines a positive evolution on $\mathcal{S}_n^d$ and has eigenvalues placed in the half plane $\mathbb{C}_-$, we may conclude that the equation (50) has an unique solution and that solution has nonnegative components. Therefore, (47) is satisfied.

Employing (36) we remark that we may rewrite (45) in the form:

$$(\mathfrak{R}'_k[0]\mathbf{X}^{(1)})(i) + M_k(i) = 0\,.$$

Applying Lemma 13 (ii) taking $\mathbf{Y} = \mathbf{X}^{(1)}$ and $\mathbf{W} = 0$ we obtain the following equation satisfied by $\mathbf{X}^{(1)}$:

$$(\mathfrak{R}'_k[\mathbf{X}^{(1)}]\mathbf{X}^{(1)})(i) - 2\mathbb{S}_i^T(\mathbf{X}^{(1)})\Pi^k(i)\mathbb{S}_i(\mathbf{X}^{(1)}) + M_k(i) = 0, 1 \le k \le N, i \in \mathcal{D}. \tag{51}$$

On the other hand, applying Lemma 13 (i) in the case of equation (49) we obtain the following equation satisfied by $\mathbf{H} \in \tilde{\mathfrak{H}} \subset \mathfrak{H}$:

$$\begin{aligned}&(\mathfrak{R}'_k[\mathbf{X}^{(1)}]\mathbf{H})(i) + \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(1)})\\ &-\mathbb{S}_i^T(\mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(1)}) + M_k(i) + \hat{M}_k(i) = 0\,.\end{aligned} \tag{52}$$

Subtracting (51) from (52) we obtain that $\mathbf{H}-\mathbf{X}^{(1)}$ satisfies the following linear equation on $\mathfrak{X}$:

$$\mathfrak{R}'[\mathbf{X}^{(1)}](\mathbf{H} - \mathbf{X}^{(1)}) + \mathbf{U}^{(1)} = 0\,, \tag{53}$$

where $\mathbf{U}^{(1)} = ((U_1^{(1)}(1), \dots, U_1^{(1)}(d)), \dots, (U_N^{(1)}(1), \dots, U_N^{(1)}(d)))$ with

$$U_k^{(1)}(i) = \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(1)}) + \mathbb{S}_i^T(\mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(1)}) + \hat{M}_k(i)\,.$$

Based on (41), (36) together with Remark 9 b), we deduce that $U_k^{(1)}(i) \succcurlyeq \hat{M}_k(i) \succ 0$.

This means that $\mathbf{U}^{(1)} \in Int\,\mathcal{X}_+$. On the other hand, from Remark 10 it follows that the linear operator $\mathfrak{R}'[\mathbf{X}^{(1)}]$ is an operator of type (19). The assumptions a), b), and c) in the statement guarantee that the assumptions of Proposition 8 are satisfied in the case of the linear operator $\mathfrak{R}'[\mathbf{X}^{(1)}]$. Therefore, the operator $\mathfrak{R}'[\mathbf{X}^{(1)}]$ defines a positive evolution on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$. Having in mind (36) and $\mathbf{U}^{(1)} \in Int\,\mathfrak{X}_+$ we obtain via Corollary 2.3.9 from [7] applied to the equation (53), that the eigenvalues of the linear operator $\mathfrak{R}'[\mathbf{X}^{(1)}]$ are in the half plane $\mathbb{C}_-$. Taking $\mathbf{X}^{(1)}$ as a first term, we construct iteratively the sequence $\{\mathbf{X}^{(p)}\}_{p\ge 1} \subset \mathfrak{X}$. More precisely, for each $p \ge 2, \mathbf{X}^{(p)}$ is obtained as a solution of the linear equation (43)-(44). We show inductively that for each $p \ge 2$, the following items hold:

a_p) $\mathbf{X}^{(p)} \preccurlyeq \mathbf{H}$ for all $\mathbf{H} \in \mathfrak{H}$;

b_p) $\mathbf{X}^{(p-1)} \preccurlyeq \mathbf{X}^{(p)}$;

c_p) the linear operator $\mathfrak{R}'[\mathbf{X}^{(p)}]$ defines a positive evolution on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$ and has the eigenvalues in the half plane $\mathbb{C}_-$.

Step 2. Consider $p = 2$. From the developments from Step 1 we know that the linear operator $\mathfrak{R}'[\mathbf{X}^{(1)}]$ defines a positive evolution on the ordered space $(\mathfrak{X}, \mathfrak{X}_+)$ and has the eigenvalues in the half plane $\mathbb{C}_-$. Under these conditions, the equation (43)-(44) written for $p = 2$ has an unique solution $\mathbf{X}^{(2)} = ((X_1^{(2)}(1), \dots, X_1^{(2)}(d)), \dots, (X_N^{(2)}(1), \dots, X_N^{(2)}(d))) \in \mathfrak{X}$. Let $\mathbf{H} \in \mathfrak{H}$ be arbitrary but fixed. Subtracting (43) (written for $p = 2$), from (52), we obtain that $\mathbf{H} - \mathbf{X}^{(2)}$ solves the linear equation on $\mathfrak{X}$:

$$\mathfrak{R}'[\mathbf{X}^{(1)}](\mathbf{H} - \mathbf{X}^{(2)}) + \mathbf{K}^{(2)} = 0\,, \tag{54}$$

where $\mathbf{K}^{(2)} = ((K_1^{(2)}(1), \ldots, K_1^{(2)}(d)), \ldots, (K_N^{(2)}(1), \ldots, K_N^{(2)}(d)))$ with

$$K_k^{(2)}(i) = \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(1)}) + \hat{M}_k(i), \tag{55}$$

$\hat{M}_k(i) \succcurlyeq 0$. From (32) and (36) we may infer that $\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(1)}) \succcurlyeq 0$. Since $\Pi^k(i) \succcurlyeq 0$ we deduce from (55) that $K_k^{(2)}(i) \succcurlyeq \hat{M}_k(i) \succcurlyeq 0$. Applying Theorem 2.3.7 from [18], we may conclude that

$$\mathbf{X}^{(2)} \preccurlyeq \mathbf{H} \tag{56}$$

whose componentwise version is:

$$X_k^{(2)}(i) \preccurlyeq H_k(i)\,, \quad 1 \le k \le N\,, \quad i \in \mathcal{D}\,.$$

Thus, we have shown that the item a_p) holds for $p = 2$. Further on, subtracting (51) from (43) written for $p = 2$, we obtain that $\mathbf{X}^{(2)} - \mathbf{X}^{(1)}$ solves the following equation

$$\mathfrak{R}'[\mathbf{X}^{(1)}](\mathbf{X}^{(2)} - \mathbf{X}^{(1)}) + \mathbf{\Delta}^{(2)} = 0\,, \tag{57}$$

where $\mathbf{\Delta}^{(2)} = ((\Delta_1^{(2)}(1), \ldots, \Delta_1^{(2)}(d)), \ldots, (\Delta_N^{(2)}(1), \ldots, \Delta_N^{(2)}(d)))$ with $\Delta_k^{(2)}(i) = \mathbb{S}_i^T(\mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(1)})$. It is obvious that $\Delta_k^{(2)}(i) \succcurlyeq 0\,, 1 \le k \le N\,, \quad i \in \mathcal{D}$. Applying Theorem 2.3.7 from [18] in the special case of equation (57) we may infer that b_p) is satisfied for $p = 2$. Applying Lemma 13 (i) in the case of equation (49) taking $\mathbf{H}, \mathbf{X}^{(2)}$ instead of $\mathbf{X}$ and $\mathbf{W}$, respectively, we obtain the following equation satisfied by $\mathbf{H} \in \tilde{\mathfrak{H}}$:

$$\begin{aligned} &(\mathfrak{R}_k'[\mathbf{X}^{(2)}]\mathbf{H})(i) + \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(2)}) \\ &-\mathbb{S}_i^T(\mathbf{X}^{(2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(2)}) + M_k(i) + \hat{M}_k(i) = 0, \quad 1 \le k \le N, i \in \mathcal{D}. \end{aligned} \tag{58}$$

On the other hand, Lemma 13 (ii) allows us to rewrite equation (43) with $p = 2$ in the form:

$$\begin{aligned} &(\mathfrak{R}_k'[\mathbf{X}^{(2)}]\mathbf{X}^{(2)})(i) - \mathbb{S}_i^T(\mathbf{X}^{(2)} - \mathbf{X}^{(1)})\Pi^k(i)\mathbb{S}_i(\mathbf{X}^{(2)}) - \\ &\mathbb{S}_i^T(\mathbf{X}^{(2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(2)} - \mathbf{X}^{(1)}) + M_k(i) - \mathbb{S}_i^T(\mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(1)}) \;=\; 0\,. \end{aligned}$$

Subtracting the last equation from (58) we get

$$\mathfrak{R}'[\mathbf{X}^{(2)}](\mathbf{H} - \mathbf{X}^{(2)}) + \mathbf{U}^{(2)} = 0\,, \tag{59}$$

where $\mathbf{U}^{(2)} = ((((U_1^{(2)}(1), \ldots, U_1^{(2)}(d)), \ldots, (U_N^{(2)}(1), \ldots, U_N^{(2)}(d)))$ with

$$\begin{aligned} &U_k^{(2)}(i) = \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(2)}) + \\ &\mathbb{S}_i^T(\mathbf{X}^{(2)} - \mathbf{X}^{(1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(2)} - \mathbf{X}^{(1)}) + \hat{M}_k(i), 1 \le k \le N, i \in \mathcal{D} \end{aligned}$$

where $\hat{M}_k(i) \succ 0$. Since a_p) and b_p) are satisfied for $p = 2$, we may deduce that $U_k^{(2)}(i) \succcurlyeq \hat{M}_k(i) \succ 0$. Based on Corollary 2.3.9 from [18] applied in the special case of equation (59) we deduce that the eigenvalues of the operator $\mathfrak{R}'[\mathbf{X}^{(2)}]$ are in the half plane $\mathbb{C}_-$. Thus, we have shown that c_p) is true for $p = 2$. In this way one sees that the equation (43) written for $p = 3$ has an unique solution and the procedure of construction of the terms of the sequence $\{\mathbf{X}^{(p)}\}_{p \ge 1}$ may continue.

Step 3. Let us assume that for a $p \geq 3$ the items a_r), b_r), c_r) are true for any $2 \leq r \leq p-1$ and we show that they still hold for $r = p$. If c_{p-1}) is fulfilled then the equation (43) has an unique solution.

Let $\mathbf{X}^{(p)} = ((X_1^{(p)}(1), \ldots, X_1^{(p)}(d)), \ldots, (X_N^{(p)}(1), \ldots, X_N^{(p)}(d)))$ be the unique solution of this equation. Consider an arbitrary $\mathbf{H} \in \mathfrak{H}$ and apply Lemma 13 (i) to rewrite the corresponding equation of type (49) as :

$$\begin{aligned}&(\mathfrak{R}'_k[\mathbf{X}^{(p-1)}]\mathbf{H})(i) - \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(p-1)})\Pi^k(i)\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(p-1)}) - \\ &\mathbb{S}_i^T(\mathbf{X}^{(p-1)})\Pi^k(i)\mathbb{S}_i(\mathbf{X}^{(p-1)}) + M_k(i) + \hat{M}_k(i) = 0, 1 \leq k \leq N, i \in \mathcal{D}.\end{aligned} \tag{60}$$

Subtracting (43) from (60) we get:

$$\mathfrak{R}'[\mathbf{X}^{(p-1)}](\mathbf{H} - \mathbf{X}^{(p)}) + \mathbf{K}^{(p)} = 0, \tag{61}$$

where $\mathbf{K}^{(p)} = ((K_1^{(p)}(1), \ldots, K_1^{(p)}(d)), \ldots, (K_N^{(p)}(1), \ldots, K_N^{(p)}(d)))$ with

$$K_k^{(p)}(i) = \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(p-1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(p-1)}) + \hat{M}_k(i)\,.$$

Since a_{p-1}) is satisfied we may deduce that $K_k^{(p)}(i) \succcurlyeq \hat{M}_k(i) \succ 0$. Hence, applying (iii) - (iv) of Theorem 2.3.7 from [18] in the special case of the equation (61) we may conclude that

$$X_k^{(p)}(i) \preccurlyeq H_k(i)\,, \quad 1 \leq k \leq N\,, \quad i \in \mathcal{D}\,, \ \text{ if } \ \mathbf{H} \in \mathfrak{H} \tag{62}$$

which confirms the validity of a_p).

Employing Lemma 13 (ii) in the case of the equation (43) (written for p replaced by $p-1$) one obtains:

$$\begin{aligned}&(\mathfrak{R}'_k[\mathbf{X}^{(p-1)}]\mathbf{X}^{(p-1)})(i) - \mathbb{S}_i^T(\mathbf{X}^{(p-2)} - \mathbf{X}^{(p-1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-1)}) \\ &+\mathbb{S}_i^T(\mathbf{X}^{(p-1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-2)} - \mathbf{X}^{(p-1)}) + M_k(i) \\ &-\mathbb{S}_i^T(\mathbf{X}^{(p-2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-2)}) = 0, 1 \leq k \leq N\,,\ i \in \mathcal{D}\,.\end{aligned} \tag{63}$$

Subtracting (63) from (43) one obtains the equation:

$$(\mathfrak{R}'[\mathbf{X}^{(p-1)}]\,(\mathbf{X}^{(p)} - \mathbf{X}^{(p-1)}) + \Delta^{(p)} = 0\,, \tag{64}$$

where $\boldsymbol{\Delta}^{(p)} = ((\Delta_1^{(p)}(1), \ldots, \Delta_1^{(p)}(d)), \ldots, (\Delta_N^{(p)}(1), \ldots, \Delta_N^{(p)}(d)))$ with $\Delta_k^{(p)}(i) = \mathbb{S}_i^T(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p-2)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p-2)})$. Since b_{p-1}) holds we deduce that $\Delta_k^{(p)}(i) \succcurlyeq 0\,, 1 \leq k \leq N\,, \quad i \in \mathcal{D}\,.$

Employing again c_{p-1}) and parts (iii) and (iv) of Theorem 2.3.7 from [18] in the case of equation (64), we may conclude that $X_k^{(p-1)}(i) \preccurlyeq X_k^{(p)}(i)\,,\ 1 \leq k \leq N\,,\ i \in \mathcal{D}$, which shows that b_p) holds. Further on, applying Lemma 13 (i) we may rewrite equation (49) satisfied by $\mathbf{H} \in \tilde{\mathfrak{H}} \subset \mathfrak{H}$ as follows:

$$\begin{aligned}&(\mathfrak{R}'_k[\mathbf{X}^{(p)}]\mathbf{H})(i) - \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(p)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(p)}) - \\ &\mathbb{S}_i^T(\mathbf{X}^{(p)})\,\Pi^{(k)}(i)\,\mathbb{S}_i(\mathbf{X}^{(p)}) + M_k(i) + \hat{M}_k(i) = 0, 1 \leq k \leq N, i \in \mathcal{D}.\end{aligned} \tag{65}$$

Applying Lemma 13 (ii) we may rewrite the equation (43) in the form :

$$\begin{aligned}&(\mathfrak{R}'_k[\mathbf{X}^{(p)}]\mathbf{X}^{(p)})(i) - \mathbb{S}_i^T(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p)}) + \\ &\mathbb{S}_i^T(\mathbf{X}^{(p)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-1)} - \mathbf{X}^{(p)}) + M_k(i) \\ &-\mathbb{S}_i^T(\mathbf{X}^{(p-1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p-1)}) = 0, 1 \le k \le N\,,\ i \in \mathcal{D}.\end{aligned}$$

Subtracting the last equation from (65) we obtain:

$$\mathfrak{R}'[\mathbf{X}^{(p)}]\,(\mathbf{H} - \mathbf{X}^{(p)}) + \mathbf{U}^{(p)} = 0, \tag{66}$$

where $\mathbf{U}^{(p)} = ((U_1^{(p)}(1), \ldots, U_1^{(p)}(d)), \ldots, (U_N^{(p)}(1), \ldots, U_N^{(p)}(d)))$ with $U_k^{(p)}(i) = \mathbb{S}_i^T(\mathbf{H} - \mathbf{X}^{(p)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \mathbf{X}^{(p)}) + \mathbb{S}_i^T(\mathbf{X}^{(p)} - \mathbf{X}^{(p-1)})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{X}^{(p)} - \mathbf{X}^{(p-1)}) + \hat{M}_k(i)\,, 1 \le k \le N\,, \quad i \in \mathcal{D}\,.$

Since a_p), b_p) are true, we obtain via Remark 9 (b) that $U_k^{(p)}(i) \succcurlyeq \hat{M}_k(i) \succ 0\,, 1 \le k \le N\,,\ i \in \mathcal{D}\,,$ which means that $\mathbf{U}^{(p)} \in Int\mathfrak{X}_+$. Applying Corollary 2.3.9 from [18] in the case of equation (66) we may conclude that c_p) holds, too.

Under these conditions one may compute $\mathbf{X}^{(p+1)}$ as an unique solution of (43) written for p replaced by $p+1$ and thus the construction of the terms of sequence $\{\mathbf{X}^{(p)}\}_{p\ge 1} \subset \mathfrak{X}$ may continue and items a_p), b_p), c_p) are true for any natural p.

Step 4. According to a_p) and b_p) it follows that the sequences $\{X_k^{(p)}(i)\}_{p\ge 1}, 1 \le k \le N\,,\ i \in \mathcal{D}$ are nondecreasing and bounded from above, hence convergent. Let $\tilde{\mathbf{X}} = ((\tilde{X}_1(1), \ldots, \tilde{X}_1(d)), \ldots, (\tilde{X}_N(1), \ldots, \tilde{X}_N(d)))$ be defined by

$$\tilde{X}_k(i) = \lim_{p\to\infty} X_k^{(p)}(i), 1 \le k \le N, i \in \mathcal{D}. \tag{67}$$

Taking the limit for $p \to \infty$ in (43) and (44), we obtain that $\tilde{\mathbf{X}}$ defined in (67) is a solution of (8) with the property that $0 \preccurlyeq \tilde{\mathbf{X}} \preccurlyeq \mathbf{H}$ for all $\mathbf{H} \in \mathfrak{H}$. Since $\mathfrak{H}$ contains any solutions of (8) $\mathbf{Y} = ((Y_1(1), \ldots, Y_1(d)), \ldots, (Y_N(1), \ldots, Y_N(d)))$ with $Y_k(i) \preccurlyeq 0\,, 1 \le k \le N\,, \quad i \in \mathcal{D}\,,$ we deduce that $\tilde{X}_k(i) \succcurlyeq Y_k(i)$ for any k and i, which confirms the minimality of $\tilde{\mathbf{X}}$ among the componentwise positive solutions of (8). In order to end the proof it remains to show that the eigenvalues of the operator $\mathfrak{R}'[\tilde{\mathbf{X}}]$ are placed in the left half plane $\mathbb{C}_-$. To this end, we choose as $\mathbf{H} \in \tilde{\mathfrak{H}}$ and rewrite the corresponding equation (49) in the form:

$$\begin{aligned}&(\mathfrak{R}'_k[\tilde{\mathbf{X}}]\mathbf{H})(i) - \mathbb{S}_i^T(\mathbf{H} - \tilde{\mathbf{X}})\,\Pi^k(i)\,\mathbb{S}_i(\mathbf{H} - \tilde{\mathbf{X}}) \\ &-\mathbb{S}_i^T(\tilde{\mathbf{X}})\,\Pi^k(i)\,\mathbb{S}_i(\tilde{\mathbf{X}}) + M_k(i) + \hat{M}_k(i) \;=\; 0\,,\end{aligned} \tag{68}$$

where $\hat{M}_k(i) \succ 0\,, 1 \le k \le N\,,\ i \in \mathcal{D}\,.$

On the other hand the equation (31) satisfied by $\tilde{\mathbf{X}}$ one may rewrite as follows:

$$(\mathfrak{R}'_k[\tilde{\mathbf{X}}]\tilde{\mathbf{X}})(i) - \mathbb{S}_i^T(\tilde{\mathbf{X}})\Pi^k(i)\mathbb{S}_i(\tilde{\mathbf{X}}) + M_k(i) = 0, 1 \le k \le N, i \in \mathcal{D}. \tag{69}$$

Subtracting (69) from (68) we obtain:

$$\mathfrak{R}[\tilde{\mathbf{X}}]\,(\mathbf{H} - \tilde{\mathbf{X}}) + \tilde{\mathbf{U}} = 0\,, \tag{70}$$

where $\tilde{\mathbf{U}} = ((\tilde{U}_1(1), \dots, \tilde{U}_1(d)), \dots, (\tilde{U}_N(1), \dots, \tilde{U}_N(d)))$ with

$$\tilde{U}_k(i) = \mathbb{S}_i^T(\mathbf{H} - \tilde{\mathbf{X}})\, \Pi^k(i)\, \mathbb{S}_i(\mathbf{H} - \tilde{\mathbf{X}}) + \hat{M}_k(i)\,, 1 \leq k \leq N\,,\ i \in \mathcal{D}.$$

Since $\mathbb{S}_i^T(\mathbf{H} - \tilde{\mathbf{X}}) \succcurlyeq 0$ and $\Pi^k(i) \succcurlyeq 0$ we may infer that $\tilde{U}_k(i) \succcurlyeq \hat{M}_k(i) \succ 0\,, 1 \leq k \leq N\,,\ i \in \mathcal{D}$, which means that $\tilde{\mathbf{U}} \in Int\mathfrak{X}_+$. We may use Corollary 2.3.9 from [18] in the case of the equation (70) to conclude that the eigenvalues of the operator $\mathfrak{R}'[\tilde{\mathbf{X}}]$ are in the half plane $\mathbb{C}_-$, which means that $\tilde{\mathbf{X}}$ defined by (67) is the strong stabilizing solution of (8). Thus the proof is complete. □

Remark 16 *a) One sees that if the assumptions from Theorem 15 are fulfilled then the assumptions of Proposition 12 are satisfied too. Hence, the solution* $\tilde{\mathbf{X}}$ *is a stabilizing solution of the SGTARE (8).*

b) The assumption that the system (42) is ESMS seems to be a restrictive condition. It could be weaked assuming, for example, that the system (1) is stabilizable with preservation of the positivity of the closed-loop system. To our knowledge there are few results refereing to the problem of stabilizability of positive systems with Markovian jumping. There are the reasons for which the problem of weaked the condition expressed by exponential stability in mean square of the system (42) remains a challenge for future research.

c) The iterative process described by (43)-(44) with the initialization provided by the solution of equation (49), which is given in the Appendix, can be used for numerical computation of the strong stabilizing solution of SGTARE (8). However, at each step $p \geq 2$*, to compute the approximation* $\mathbf{X}^{(p)}$ *we need to solve a linear equation of high dimension.*

In the next section we shall prove the practical implementation of the process where the Nash equilibrium is established of the infinite horizon linear quadratic differential games for Markovian jump for linear positive systems.

5 The practical realization

We can rewrite (37) using (38) in the form

$$\begin{aligned}(\mathfrak{R}'_k[\mathbf{X}]\mathbf{Y})\,(i) \ &= \ (\Gamma(i) + \tfrac{1}{2}q_{ii}I_n)^T Y_k(i) + Y_k(i)(\Gamma(i) + \tfrac{1}{2}q_{ii}I_n) \\ &\quad + \textstyle\sum_{j=1, j\neq i}^{d} q_{ij} Y_k(j) + \sum_{l=1, l\neq k}^{N} [\Xi_{kl}(i)^T Y_l(i) + Y_l(i)\Xi_{kl}(i)].\end{aligned} \tag{71}$$

Thus, iteration (43) becomes:

$$\begin{aligned}0 \ &= \ (\Gamma_k^{(p-1)}(i))^T X_k^{(p)}(i) + X_k^{(p)}(i)\,\Gamma_k^{(p-1)}(i) \\ &\quad + \textstyle\sum_{l=1, l\neq k}^{N} [(\Xi_{kl}^{(p-1)}(i))^T X_l^{(p)}(i) + X_l^{(p)}(i)\Xi_{kl}^{(p-1)}(i)] \\ &\quad + \textstyle\sum_{j=1, j\neq i}^{d} q_{ij} X_k^{(p)}(j) + \mathbf{\Phi}^{(p-1)}\end{aligned} \tag{72}$$

with

$$\begin{cases} \Gamma_k^{(p-1)}(i) = A(i) - \sum_{l=1}^{N} S_l(i)\, X_l^{(p-1)}(i) + \frac{1}{2}q_{ii}I_n\,, \\ \Xi_{kl}^{(p-1)}(i) = S_{kl}(i)X_l^{(p-1)}(i) - S_l(i)X_k^{(p-1)}(i). \end{cases}$$

We consider a standard natural dynamic interpretation to the way to choose the best strategy of each player. We present a learning process in which each player refines its strategy by observing the actual choice of strategies of the other players. The learning process converges because of Theorem 15. The realization of the game is presented as follows. ach player starts the game computing the unique solution of the set of generalized Lyapunov equations (GLE) (45). The solution $\mathbb{X}_k^{(1)}$ of (45) has the property that the eigenvalues of the linear operator $\mathcal{R}'_k[\mathbb{X}^{(1)}]$ are in the half plane $\mathbb{C}_-$ and moreover the eigenvalues of the linear operator $\mathcal{R}'[\mathbf{X}^{(1)}]$ are in the half plane $\mathbb{C}_-$ (see Step 1 of the proof of the Theorem 15). Each player presents his new strategy $F_k^{(1)}(i) = -R_{kk}^{-1}(i)B_k^T(i)\,X_k^{(1)}(i)\,,\ \ 1 \le k \le N\,,\ i \in \mathcal{D}\,,$ which is obtained knowing $\mathbb{X}_k^{(1)}, 1 \le k \le N$. Thus, the other players have the possibility to update his own behavior based on the information for $(\,\mathbb{F}_1^{(1)}, \mathbb{F}_2^{(1)}, \mathbb{F}_3^{(1)}\,)$.

At the p-th step, the player with number k guesses that the other players will preserve their strategies from the previous step, i.e. $\mathbb{X}_1^{(p-1)}, \ldots, \mathbb{X}_{k-1}^{(p-1)}, \mathbb{X}_{k+1}^{(p-1)}, \ldots, \mathbb{X}_N^{(p-1)}$. Thus, the player k considers equation (72) as a reaction curve. Knowing matrices $\mathbb{X}_1^{(p-1)}, \ldots, \mathbb{X}_{k-1}^{(p-1)}$, $\mathbb{X}_{k+1}^{(p-1)}, \ldots, \mathbb{X}_N^{(p-1)}$ he can put them in (72) instead of $X_l^{(p)}(i), (l \ne k)$ and thus, the following recurrence equation is obtained:

$$0 \;=\; (\Gamma_k^{(p-1)}(i))^T X_k^{(p)}(i) + X_k^{(p)}(i)\,\Gamma_k^{(p-1)}(i) + \textstyle\sum_{j=1, j\ne i}^d q_{ij} X_k^{(p)}(j) + \tilde{\mathbf{\Phi}}_k^{(p-1)} \qquad (73)$$

with

$$\tilde{\mathbf{\Phi}}_k^{(p-1)} = \mathbf{\Phi}^{(p-1)} + \sum_{l=1, l\ne k}^{N} [(\Xi_{kl}^{(p-1)}(i))^T \underline{X_l^{(p-1)}(i)} + \underline{X_l^{(p-1)}(i)}\Xi_{kl}^{(p-1)}(i)].$$

The player k solves a set of linear matrix equations (73). As a result the unique solution $\mathbb{X}_k^{(p)}$ of (73) is obtained. The learning process is continued and moreover the strong stabilizing solution $\tilde{\mathbf{X}}$ of (8) is computed, which defines the feedback Nash equilibrium $(\mathbb{F}_1, \ldots, \mathbb{F}_N)$ (see Definition 5). Further on, we consider the following generalized Lyapunov equations:

$$\begin{aligned} &\mathcal{G}L_i(\mathbf{Z}) \;:=\; E(i)^T\, Z_k(i) + Z_k(i)\, E(i) + \textstyle\sum_{j=1}^d q_{ij} Z_k(j) + Q_k(i) = 0\,, \\ &i \in \mathcal{D},\ 1 \le k \le N\,. \end{aligned} \qquad (74)$$

We introduce an algorithm to compute the solution of (74).

Algorithm SGLE (Set of Generalized Lyapunov Equations).

1. We fix $k,\ 1 \le k \le N$ and $tol = 1e-7$. We take $Z_k^{(0)}(i) = 0, i \in \mathcal{D}$.
2. For $s = 0, \ldots$ we solve the Lyapunov equation

$$(E(i) + \frac{1}{2}q_{ii}I_n)^T\, Z_k^{(s+1)}(i) + Z_k^{(s+1)}(i)\,(E(i) + \frac{1}{2}q_{ii}I_n) + \tilde{Q}_k(i) = 0$$

with $\tilde{Q}_k(i) = Q_k(i) + \sum_{j=1, j\ne i}^d q_{ij} Z_k^{(s)}(j)$.

3. until $\max_i\ \|\mathcal{G}L_i(\mathbf{Z}^{(s+1)})\| \le tol$. The matrix $\mathbf{Z}^{(s+1)}$ is the solution of (74). (see [20])

Thus we describe the following algorithm to realize the above learning process to compute the strong stabilizing solution to (8).

Algorithm FNE (Feedback Nash Equilibrium).

1. There are N players. The player $k(1 \leq k \leq N)$ solves the set of equations (45) applying the **Algorithm SGLE**. The stabilizing solution $\mathbb{X}_k^{(1)} = \mathbf{Z}^{(s+1)}$ is computed.

2. For p=2, ... the following is executed:

2.1. Each player solves a set of linear matrix equations (73) via **Algorithm SGLE**. The stabilizing solution $\mathbb{X}_k^{(p)} = \mathbf{Z}^{(s+1)}$ is computed.

2.2. until $\max_{k,i} \|\mathfrak{R}_k[\mathbf{X}^{(p)}](i)\|_2 \leq \varepsilon$, where $\mathfrak{R}_k[\mathbf{X}](i)$ is defined in (32). We take $\tilde{\mathbf{X}} = \mathbf{X}^{(p)}$.

Example. We consider a three players game ($N = 3$), the size of the finite set $\mathcal{D}$ is $d = 4$. The matrix Q has entries $(q_{ij}) = \begin{pmatrix} -0.85 & 0.12 & 0.48 & 0.25 \\ 0.15 & -0.95 & 0.24 & 0.56 \\ 0.34 & 0.44 & -0.93 & 0.15 \\ 0.18 & 0.28 & 0.42 & -0.88 \end{pmatrix}$. We take $n = 4, m_1 = m_2 = m_3 = 3$. We define all matrix coefficients using the Matlab description. We choose matrix coefficients $A(i) \in \mathbb{R}^{n\times n}$ and $B_1(i), B_2(i), B_3(i) \in \mathbb{R}^{4\times 3}$ of the system (1) as follows ($k = 1, 2, 3$; $i = 1, 2, 3, 4$):

$$A(i) = abs(randn(n, n))/2 - 4 * eye(n, n);,$$
$$B_k(i) = full(abs(sprandn(n, 3, 0.7))/0.5);,$$

Matrix coefficients for the first player are :

$$R_{11}(1) = [-420\ 5\ 0; 5\ -400\ 0; 0\ 0\ -50];,, \quad R_{11}(2) = [-480\ 0\ 0; 0\ -460\ 12; 0\ 12\ -480];,$$
$$R_{11}(3) = [-520\ 0\ 0; 0\ -400\ 45; 0\ 45\ -60];, \quad R_{11}(4) = [-550\ 15\ 0; 15\ -380\ 25; 0\ 25\ -80];$$

$$R_{12}(1) = [40\ 0\ 0; 0\ 200\ 0; 0\ 0\ 50];, \quad R_{12}(2) = [60\ 0\ 0; 0\ 180\ 0; 0\ 0\ 20];,$$
$$R_{12}(3) = [50\ 0\ 0; 0\ 150\ 0; 0\ 0\ 40];, \quad R_{12}(4) = [55\ 0\ 0; 0\ 220\ 0; 0\ 0\ 50];,$$

$$R_{13}(1) = [50\ 0\ 0; 0\ 150\ 0; 0\ 0\ 40];, \quad R_{13}(2) = [55\ 0\ 0; 0\ 220\ 0; 0\ 0\ 50];$$
$$R_{13}(3) = [18\ 0\ 0; 0\ 140\ 0; 0\ 0\ 70];, \quad R_{13}(4) = [55\ 0\ 0; 0\ 160\ 0; 0\ 0\ 60];,$$

$$M_1(1) = 0.8 * eye(n, n);, \quad M_1(2) = 0.9 * eye(n, n);,$$
$$M_1(3) = 0.45 * eye(n, n);, \quad M_1(4) = 0.75 * eye(n, n);\ .$$

Matrix coefficients for the second player are :

$$R_{22}(1) = [-450\ 5\ 0; 5\ -300\ 0; 0\ 0\ -50];, \quad R_{22}(2) = [-480\ 0\ 18; 0\ -514\ 0; 18\ 0\ -40];,$$
$$R_{22}(3) = [-520\ 0\ 0; 0\ -300\ 15; 0\ 15\ -60];, \quad R_{22}(4) = [-475\ 0\ 0; 0\ -399\ 0; 0\ 0\ -120];,$$

$$R_{21}(1) = [44\ 0\ 0; 0\ 160\ 0; 0\ 0\ 40];, \quad R_{21}(2) = [25\ 15\ 0; 15\ 190\ 0; 0\ 0\ 90];,$$
$$R_{21}(3) = [33\ 0\ 0; 0\ 120\ 0; 0\ 0\ 40];, \quad R_{21}(4) = [20\ 0\ 10; 0\ 180\ 0; 10\ 0\ 330];,$$

$$R_{23}(1) = [44\ 0\ 0; 0\ 160\ 0; 0\ 0\ 40];, \quad R_{23}(2) = [48\ 0\ 0; 0\ 180\ 0; 0\ 0\ 50];,$$
$$R_{23}(3) = [33\ 0\ 0; 0\ 120\ 0; 0\ 0\ 40];, \quad R_{23}(4) = [25\ 15\ 0; 15\ 190\ 0; 0\ 0\ 90];,$$

$$M_2(1) = 0.25 * eye(n, n);, \quad M_2(2) = 0.95 * eye(n, n);,$$
$$M_2(3) = 0.95 * eye(n, n);, \quad M_2(4) = 1.05 * eye(n, n);\ .$$

Table 1: Results for the Example.

	The First player	The Second player	The Third player
Step 1	It=20	It=20	It=20
Step 2	It=16	It=16	It=16
Step 3	It=14	It=13	It=14
Step 4	It=11	It=11	It=11
Step 5	It=9	It=9	It=9
Step 6	It=7	It=7	It=7
norm	$\max_{1,i} \|\mathfrak{R}_1[\mathbf{X}^{(6)}](i)\|_2 =$ $= 9.2977 \times 10^{-10}$	$\max_{2,i} \|\mathfrak{R}_2[\mathbf{X}^{(6)}](i)\|_2 =$ $= 8.6521 \times 10^{-10}$	$\max_{3,i} \|\mathfrak{R}_3[\mathbf{X}^{(6)}](i)\|_2 =$ $= 6.5653 \times 10^{-10}$

Matrix coefficients for the third player are :

$$R_{33}(1) = [-350\ 8\ 0; 8\ \ -200\ 12; 0\ 12\ \ -35];,\quad R_{33}(2) = [-370\ 0\ 18; 0\ \ -114\ 0; 18\ 0\ \ -420];,$$
$$R_{33}(3) = [-320\ 0\ 0; 0\ \ -240\ 18; 0\ 18\ \ -60];,\quad R_{33}(4) = [-395\ 0\ 0; 0\ \ -120\ 0; 0\ 0\ \ -80];,$$

$$R_{31}(1) = [48\ 0\ 0; 0\ 240\ 0; 0\ 0\ 70];,\quad R_{31}(2) = [55\ 0\ 0; 0\ 260\ 0; 0\ 0\ 60];,$$
$$R_{31}(3) = [66\ 0\ 0; 0\ 250\ 0; 0\ 0\ 40];,\quad R_{31}(4) = [77\ 0\ 0; 0\ 190\ 0; 0\ 0\ 50];,$$

$$R_{32}(1) = [48\ 0\ 0; 0\ 240\ 0; 0\ 0\ 50];,\quad R_{32}(2) = [33\ 0\ 0; 0\ 120\ 0; 0\ 0\ 40];,$$
$$R_{32}(3) = [66\ 0\ 0; 0\ 280\ 0; 0\ 0\ 40];,\quad R_{32}(4) = [77\ 0\ 0; 0\ 290\ 0; 0\ 0\ 70];,$$

$$M_3(1) = 0.85 * eye(n, n);,\quad M_3(2) = 1.05 * eye(n, n);,$$
$$M_3(3) = 0.9 * eye(n, n);,\quad M_3(4) = 0.99 * eye(n, n);\ .$$

Next Table 1 shows the number of steps for which the learning process converges. They are 6 for the considered example. Moreover, each player makes any number of iterations in order to compute his best response at each step - see columns of the table.

6 Conclusions

In this paper, we have studied the problem of finding the Nash equilibrium of the infinite horizon linear quadratic differential games for positive linear systems with Markovian jumping. This problem is closely related to the problem of the existence and numerical computation of the stabilizing nonnegative solution of a set of algebraic Riccati equations. We provide a set of conditions which guarantee the existence of the stabilizing solution. We have proposed an iterative algorithm and proved in Theorem 15 that it is globally convergent.

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