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A Stochastic Linear Quadratic Optimization Problem with Sampled Measurements

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Abstract. A linear quadratic optimal control problem for the sampled-data stochastic systems is investigated. In particular, since the considered control can only access the measurement states governed by the discrete-time instances, the combination for both continuous and discrete time methodology are introduced for the first time. It is shown that the Riccati differential and difference equations play an important role in establishing the sampled data control. By using the fact that a jump system covers a sampled-data system, the results for jump stochastic systems are applied to the sampled-data systems. Finally, in order to show the effectiveness, a practical example is demonstrated.

Key Words: stochastic system, sampled-data system.

1 Introduction

The sampled-data systems have scored a great success in the past decades. In [3, 4], H_2 and linear quadratic (LQ) robust sampled-data control problems under a unified framework have been considered. The problems of stochastic stability and robust control for a class of uncertain sampled-data systems with random jumping parameters described by finite state semi-Markov process are studied in [2], where the design procedure for robust multirate sampled-data control is formulated as linear matrix inequalities.

The linear quadratic optimal control problem of sampled-data is considered for linear stochastic continuous-time systems with unknown parameters and disturbances in [5]. In [6], the optimal stochastic control problems for jump systems with sampled inputs and sampled observations has been addressed. However, the case with state and control multiplicative white noise is still open. When the practical control system is dealt with, since there exist various mixture noise, such problem is more suitable and it is natural for designing the digital controllers.

This paper is concerned with a linear quadratic optimal control problem for the sampled-data systems. It should be noted that the measurement states are governed by the discrete-time instances. Therefore, the considered control can only access the sample data. From this feature, this optimal control problem is be named LQ optimization problem with sampled measurements. In order to avoid the difficulty based on the sample measurement, the jump system methodology that the continuous and discrete time methodologies can be covered is introduced for the stochastic case. Namely, by using the useful fact that a jump system is a general class that covers continuous-time, discrete-time and sampled-data systems, the results for jump stochastic systems are introduced for the first time. As a result, the proposed control law can be computed by solving the Riccati differential and difference equations. As another important contribution of this paper is that the more general weighted matrices in the cost functional is studied. Finally, a practical example is solved to demonstrate the efficiency of the proposed controller.

2 PROBLEM FORMULATION

Let us consider the optimal control problem described by the stochastic linear controlled system:

$$\begin{aligned} dx(t) = [A_0(t)x(t) + B_0(t)u(t)]dt + [A_1(t)x(t) \\ +B_1(t)u(t)]dw(t) + B_v(t)dv(t),\ x(0) = x_0\,, \end{aligned} \tag{1}$$

and the cost functional

$$J(u, x_0) = \mathbb{E}\big[x_u^T(\tau)Gx_u(\tau)\big] + \mathbb{E}\left[\int_0^\tau \Big(x_u^T(t)M(t)x_u(t) + u^T(t)R(t)u(t)\Big)dt\right] \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the vector of control parameters at instance time t, and $\{w(t)\}_{t\geq 0}, \{v(t)\}_{t\geq 0}$ are standard Wiener processes, defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

In (2), $x_u(\cdot)$ is the solution of the problem with given initial values (1) corresponding to the input $u(\cdot)$. Throughout the paper $\mathbb{E}[\cdot]$ denotes the mathematical expectation and the superscript T stands to the transpose of a matrix or a vector. For each $t \geq 0, \mathcal{F}_t \subset \mathcal{F}$ is the σ-algebra generated by $w(s), v(s),\ 0 \leq s \leq t$.

Let $0 = t_0 < t_1 < t_2 \ldots < t_N = \tau$ be a partition of the interval $[0, \tau]$. The class of the admissible controls $\mathcal{U}_{ad}$ consists of all piecewise constants stochastic processes which are adapted to the filtration $\mathcal{F}_t, t \geq 0$. More precisely, $\mathcal{U}_{ad} = \{u : [0, \tau] \times \Omega \to \mathbb{R}^m | u(t) = u_k$, $t_k \leq t < t_{k+1}$, u_k is $\mathcal{F}_{t_k}$-measserable and $\mathbb{E}[|u_k|^2] < \infty$, $0 \leq k \leq N-1\}$.

Remark 1 *One of the admissible controls is*

$$u(t) = F(k)x(t_k), t_k \leq t < t_{k+1},\ 0 \leq k \leq N-1. \tag{3}$$

One sees that for the implementation of the controls of type (3) we need to measure the states $x(t)$ only to discrete-time instances $t_k, 0 \leq k \leq N-1$.

The optimal control problem that we want to solve can be stated as follows:

Problem 1.: Given $x_0 \in \mathbb{R}^n$, find a control $\tilde{u} \in \mathcal{U}_{ad}$ which minimizes the cost (2) over the class of the admissible controls, that is

$$J(\tilde{u}, x_0) = \min_{u \in \mathcal{U}_{ad}} J(u, x_0). \tag{4}$$

In the developments from this work we shall provide conditions which guarantee the existence of a control $\tilde{u}$ that satisfies the optimality condition (4). We shall see that the optimal control $\tilde{u}$, if any, is of type (3). From these reasons the optimal control problem stated before will be named *"linear quadratic (LQ) optimization problem with sampled measurements"*.

We shall provide explicit formulae of the gain matrices $\tilde{F}_k$ of the optimal control. In the special case $A_1(t) \equiv 0, B_1(t) \equiv 0$ this optimal control problem was solved in [6]. To solve the optimal control problem stated above, we shall use a similar method to the one used in the above mentioned reference. To this end, we shall transform the considered LQ optimization problem in an equivalent one, but associated to a stochastic controlled system with finite jumps.

The developments in this work are done under the following assumptions regarding the stochastic processes $\{w(t)\}_{t\geq 0}, \{v(t)\}_{t\geq 0}$ and the matrix coefficients in (1) and (2):

$\mathbf{H_1}$) (i) $\{w(t)\}_{t\geq 0}$ is an 1-dimensional standard Wiener process with zero mean and $\mathbb{E}[(w(t)-w(s))^2] = |t-s|$ for all $t, s \geq 0$;

(ii) $\{v(t)\}_{t\geq 0}$ is an m_v-dimensional standard Wiener process with zero mean and $\mathbb{E}[(v(t)-v(s))(v(t)-v(s))^T] = V|t-s|$ for all $t, s \geq 0$, where $V \geq 0$ is a known matrix;

(iii) $\{w(t)\}_{t\geq 0}, \{v(t)\}_{t\geq 0}$ are independent stochastic processes.

$\mathbf{H_2}$) $(A_k(\cdot), B_k(\cdot)) : [0,\tau] \to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}, k = 1,2, (B_v(\cdot), M(\cdot), R(\cdot)) : [0,\tau] \to \mathbb{R}^{n\times m_v} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times m}$ are continuous matrix valued functions.

$\mathbf{H_3}$) The weights $G, M(t), R(t)$ are in one of the following cases:

a) $M(t) = M^T(t) \geq 0,\ R(t) = R^T(t) > 0,\ t \in [0,\tau],\ \ G = G^T \geq 0$.

b) $M(t) = 0,\ R(t) = 0,\ t \in [0,\tau], G = G^T \geq 0$.

Remark 2 *a) The conditions in assumption* $\mathbf{H_3}$ *a) are similar with the usual ones in the classical problem of linear quadratic regulator in the deterministic and stochastic framework. The optimal control problem asking for the minimization of the cost (2) on the set of admissible controls* $\mathcal{U}_{ad}$ *may be solved in a more general setting where the weights matrices* $M(\cdot), R(\cdot)$ *and* G *are with indefinite sign, but this exceed the goal of this work.*

b) In the case when the weights satisfy assumption $\mathbf{H_3}$ *b), the performance criterion (2) reduced to:*

$$J(u, x_0) = \mathbb{E}[x_u^T(\tau) G x_u(\tau)]. \tag{5}$$

In this special case, the aim of the optimal control is to minimize the mean square of the value of a suitable output of the controlled system (1) at instance time τ.

If $u \in \mathcal{U}_{ad}$, system (1) takes the form

$$\begin{aligned} dx(t) &= [A_0(t)x(t) + B_0(t)u_k])dt + [A_1(t)x(t) \\ &\quad + B_1(t)u_k]dw(t) + B_v(t)dv(t), \end{aligned} \tag{6}$$

$t_k \leq t < t_{k+1},\ 0 \leq k \leq N-1, x(0) = x_0$ and the cost functional (2) takes the form:

$$\begin{aligned} J(u, x_0) &= \mathbb{E}[x_u^T(\tau) G x_u(\tau)] + \mathbb{E}\left[\int_0^\tau x_u^T(t) M(t) x_u(t) dt\right] \\ &+\mathbb{E}\left[\sum_{k=0}^{N-1} u_k^T \mathcal{R}(k) u_k\right]. \end{aligned}$$

Setting $\xi(t) = (x^T(t) \quad u^T(t))^T$, we rewrite (6) in the form

$$\begin{aligned} d\xi(t) &= \mathcal{A}_0(t)\xi(t)dt + \mathcal{A}_1(t)\xi(t)dw(t) + \mathcal{B}_v(t)dv(t), \\ &\quad t_k < t \leq t_{k+1}, \\ \xi(t_k^+) &= \mathcal{A}_d \xi(t_k) + \mathcal{B}_d u_k, \\ &\quad k = 0,\ 1, \ldots, N-1,\ \xi(0) = \xi_0 = (x_0^T \quad u_0^T)^T, \end{aligned} \tag{7}$$

where $k = 1,\ 2,\ p = n + m$,

$$\begin{aligned} \mathcal{A}_k &= \begin{pmatrix} A_k & B_k \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times p},\ \mathcal{B}_v = \begin{pmatrix} B_v(t) \\ 0 \end{pmatrix} \in \mathbb{R}^{p \times m_v}, \\ \mathcal{A}_d &= \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times p},\ \mathcal{B}_d = \begin{pmatrix} 0 \\ I_m \end{pmatrix} \in \mathbb{R}^{p \times m}. \end{aligned} \tag{8}$$

The cost functional (7) may be rewritten in the form:

$$\begin{aligned} J_d(\mathbf{u}, x_0) &= \mathbb{E}[\xi^T(\tau)\mathcal{G}\xi(\tau)] + \mathbb{E}\left[\int_0^\tau (\xi^T(t)\mathcal{M}(t)\xi(t))dt\right] \\ &+\mathbb{E}\left[\sum_{k=0}^{N-1} u_k^T \mathcal{R}(k) u_k\right], \end{aligned}$$

for all $\mathbf{u} = \{u_0, u_1, \ldots, u_{N-1}\}, u_k \in \mathbb{R}^m$, where

$$\begin{aligned} &\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times p},\ \mathcal{M}(t) = \begin{pmatrix} M(t) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times p}, \\ &\mathcal{R}(k) = \int_{t_k}^{t_{k+1}} R(t)dt,\ k = 0,\ 1, \ldots, N-1. \end{aligned} \tag{9}$$

The set of admissible controls $\mathcal{U}_{ad}^d$ of the optimal control problem described by the system with finite jumps (7) and the cost functional (9) consists of all finite sequences of m-dimensional random vectors $\mathbf{u} = \{u_0, u_1, \ldots, u_{N-1}\}$ with the property that for each k, u_k are $\mathfrak{F}_{t_k}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$.

One sees that there exists a one to one corespondence between the set of admissible controls $\mathcal{U}_{ad}$ and the set of admissible controls $\mathcal{U}_{ad}^d$, and we have:

$$inf_{u \in \mathcal{U}_{ad}}\{J(u, x_0)\} = \inf_{\mathbf{u} \in \mathcal{U}_{ad}^d} J_d(\mathbf{u}, x_0). \tag{10}$$

This shows that there exists an optimal control $\tilde{u} \in \mathcal{U}_{ad}$ minimizing the cost functional (2) over the set of admissible piecewise constant controls $\mathcal{U}_{ad}$ if and only if there exists an optimal control $\tilde{\mathbf{u}} \in \mathcal{U}_{ad}^d$, minimizing the cost functional (9) over the set of admissible controls $\mathcal{U}_{ad}^d$.

The control $\tilde{\mathbf{u}}$ that minimizes the cost functional (9) over the set of admissible controls $\mathcal{U}_{ad}^d$ is obtained applying a more general result derived in the next section.

3 A LQ control for finite jumps case

Let us consider the controlled system with finite jumps:

$$\begin{aligned} d\xi(t) &= \hat{A}_0(t)\xi(t)dt + \hat{A}_1(t)\xi(t)dw(t) \\ &\quad +\hat{B}_v(t)dv(t), \quad t_k < t \leq t_{k+1}, \\ \xi(t_k^+) &= \hat{A}_d(k)\xi(t_k) + \hat{B}_d(k)u_k, 0 \leq k \leq N-1, \\ \xi(0) &= \xi_0, \; 0 = t_0 < t_1 < \ldots < t_N = \tau, \end{aligned} \tag{11}$$

being a partition of the interval $[0,\tau]$; $\xi(t) \in \mathbb{R}^{\hat{n}}$ is the state vector, $u_k \in \mathbb{R}^m$ are the control parameters and $\{w(t)\}_{t\geq 0}, \{v(t)\}_{t\geq 0}$ are stochastic processes satisfying the assumption $\mathbf{H_1}$. In (11) $\hat{A}_k(\cdot) : [0,\tau] \to \mathbb{R}^{\hat{n}\times\hat{n}}, k = 1,2$ $\hat{B}_v(\cdot) : [0,\tau] \to \mathbb{R}^{\hat{n}\times m_v}$ are continuous matrix valued functions and $\hat{A}_d(k) \in \mathbb{R}^{\hat{n}\times\hat{n}}, \hat{B}_d(k) \in \mathbb{R}^{\hat{n}\times m}, k = 0, \; 1,\ldots,N-1$, are known matrices. As in the previous section, $\mathcal{U}_{ad}^d$ is the set of all finite sequences of m-dimensional random vectors $\mathbf{u} = \{u_0, u_1, \ldots, u_{N-1}\}$ with the property that for each k , u_k are $\mathfrak{F}_{t_k}$-measurable and $\mathbb{E}[|u_k|^2] < \infty$. Let us consider the quadratic performance criterion $J_d : \mathcal{U}_{ad}^d \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}_d(\mathbf{u},\xi_0) &= \mathbb{E}[\xi^T(\tau)\hat{G}\xi(\tau)] + \mathbb{E}\left[\int_0^\tau \xi_{\mathbf{u}}^T(t)\hat{M}(t)\xi_{\mathbf{u}}(t)dt\right] \\ &\quad +\mathbb{E}\left[\sum_{k=0}^{N-1} u_k^T\hat{R}(k)u_k\right], \end{aligned} \tag{12}$$

where $\xi_{\mathbf{u}}(t)$ is the solution of the problem with given initial values (11) determined by the control $\mathbf{u} = \{u_0, u_1, \ldots, u_{N-1}\} \in \mathcal{U}_{ad}^d$.

In (12) $\hat{M}(\cdot) : [0,\tau] \to \mathcal{S}_{\hat{n}}$ is a continuous matrix valued function and $\hat{G} \in \mathcal{S}_{\hat{n}}, \hat{R}(k) \in \mathcal{S}_m, 0 \leq k \leq N-1$.

Throughout the paper $\mathcal{S}_d$ stands for the linear space of $d \times d$ symmetric matrices. Applying Theorem 5.2.1 from [7] on each interval $[t_k, t_{k+1}]$ we obtain:

Proposition 3 *For each $\mathbf{u} \in \mathcal{U}_{ad}^d$ and $\xi_0 \in \mathbb{R}^{\hat{n}}$, the problem with given initial values (11) has a unique solution $\xi_{\mathbf{u}} : [0,\tau] \to \mathbb{R}^{\hat{n}}$ having the properties:*
(i) *$t \to \xi_{\mathbf{u}}(t)$ is left continuous a.s. in any $t_0 \in [0,\tau]$;*
(ii) *for each $0 < t \leq \tau, \xi_{\mathbf{u}}$ is $\mathfrak{F}_t$-measurable;*
(iii) *$E[|\xi_{\mathbf{u}}(t)|^2] < \infty$ for all $t \in [0,\tau]$;*
(iv) *$\xi(0) = \xi_0$.*

The optimal control problem that we want to solve in this section can be stated as follows:

Problem 2. Given $\xi_0 \in \mathbb{R}^{\hat{n}}$ find a control $\tilde{\mathbf{u}} \in \mathcal{U}_{ad}^d$ with the property that $\mathcal{J}_d(\tilde{\mathbf{u}},\xi_0) \leq \mathcal{J}_d(\mathbf{u},\xi_0)$ for any $\mathbf{u} \in \mathcal{U}_{ad}^d$.

Based on the coefficients of the system (11) and the cost functional (12), we introduce the following backward differential align with finite jumps of Riccati type:

$$\begin{aligned}
-\dot{X}(t) &= \hat{A}_0^T(t)X(t) + X(t)\hat{A}_0(t) + \hat{A}_1^T(t)X(t)\hat{A}_1(t) + \hat{M}(t) \\
&\quad t_k \le t < t_{k+1}, \\
X(t_k^-) &= \hat{A}_d^T(k)X(t_k)\hat{A}_d(k) - \hat{A}_d^T(k)X(t_k)\hat{B}_d(k) \\
&\quad \times \Big(\hat{R}(k) + \hat{B}_d^T(k)X(t_k)\hat{B}_d(k)\Big)^\dagger \hat{B}_d^T(k)X(t_k)\hat{A}_d(k) \\
&\quad k = 0, \ldots, N-1.
\end{aligned} \tag{13}$$

Here and after, $\Big(\hat{R}(k) + \hat{B}_d^T(k)X(t_k)\hat{B}_d(k)\Big)^\dagger$ denotes the pseudo-inverse of the matrix $\Big(\hat{R}(k) + \hat{B}_d^T(k)X(t_k)\hat{B}_d(k)\Big)$. For definition and some properties of the pseudo-inverse of a matrix we refer to [8], see also [9].

We introduce the following assumption regarding the differential align with finite jumps of Riccati type (13).

$\mathbf{H_4}$) The backward differential align with finite jumps of Riccati type (13) has a unique solution $\hat{X}(\cdot) : [0, \tau] \to \mathcal{S}_{\hat{n}}$ satisfying the terminal value $\hat{X}(\tau) = \hat{G}$ and has the following properties for all $0 \le k \le N-1$:

$$\hat{R}(k) + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k) \ge 0, \tag{14}$$

$$\begin{aligned}
&\Big[I_m - (\hat{R}(k) + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k))^\dagger\big(\hat{R}(k) \\
&\quad + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k)\big)\Big]\hat{B}_d^T(k)\hat{X}(t_k)\hat{A}_d(k) = 0
\end{aligned} \tag{15}$$

Let us introduce the notations: $\Pi(k) = \hat{R}(k) + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k)$, $\Lambda(k) = \hat{B}_d^T(k)\hat{X}(t_k)\hat{A}_d(k)$. Thus (14) becomes $\Pi(k) \ge 0$ and (15) is rewritten as: $[I_m - \Pi(k)^\dagger\Pi(k)]\Lambda(k) = 0$, $k = 0,\ 1, \ldots N-1$.

The main result of this section is:

Theorem 4 *Assume that the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_4}$ *are fulfilled. Consider controls of the form:*

$$\mathbf{u}_{\psi\varphi} = \{u_{\psi\varphi 0}, u_{\psi\varphi 1}, \ldots, u_{\psi\varphi N-1}\}, \tag{16}$$

where $u_{\psi\varphi k} = -[\Pi(k)^\dagger\Lambda(k) + [I_m - \Pi(k)^\dagger\Pi(k)]\Psi(k)]\tilde{\xi}(t_k) - [I_m - \Pi(k)^\dagger\Pi(k)]\varphi(k)$. *Furthermore,* $\Psi(k) \in \mathbb{R}^{m\times\hat{n}}$, $\varphi(k) \in \mathbb{R}^m$, $0 \le k \le N-1$ *are arbitrary, and* $\tilde{\xi}(t_k)$ *are the values of the solution of the system obtained when (16) is plugged in (11). Under the considered assumptions the controls defined by (16) lie in* $\mathcal{U}_{ad}^d$ *and they minimize the cost functional (12) over the set of admissible controls* $\mathcal{U}_{ad}^d$. *The minimal value of the cost (12) is given by*

$$\begin{aligned}
\mathcal{J}_d(\mathbf{u}_{\psi\varphi}, \xi_0) &= \xi_0^T\hat{X}(0^-)\xi_0 \\
&\quad + \int_0^\tau Tr\hat{X}(t)\hat{B}_v(t)V\hat{B}_v^T(t)]dt.
\end{aligned} \tag{17}$$

Proof: Applying the Ito formula on an interval $[s_1, s_2] \subset [t_k, t_{k+1}]$ for the function $\xi^T \hat{X}(t)\xi$ and letting $s_1 \to t_k$ and $s_2 \to t_{k+1}$ one obtains via the first align from (13), that:

$$\mathbb{E}[\int_{t_k}^{t_{k+1}} \xi^T(t)\hat{M}(t)\xi(t)dt] + \mathbb{E}[\xi^T(t_{k+1})\hat{X}(t_{k+1}^-)\xi(t_{k+1})] \\ - \mathbb{E}[\xi^T(t_k^+)\hat{X}(t_k)\xi(t_k^+)] = \int_{t_k}^{t_{k+1}} \mathbf{Tr}[\hat{X}(t)\hat{B}_v(t)V\hat{B}_v^T(t)]dt.$$

Employing the second align of (11) we further obtain

$$\mathbb{E}\left[\int_{t_k}^{t_{k+1}} \xi^T(t)\hat{M}(t)\xi(t)dt\right] + \mathbb{E}[u_k^T(t)\hat{R}(k)u_k] \\ + \mathbb{E}[\xi^T(t_{k+1})\hat{X}(t_{k+1}^-)\xi(t_{k+1})] \\ - \mathbb{E}[\xi^T(t_k)\hat{A}_d^T(k)\hat{X}(t_k)\hat{A}_d(k)\xi(t_k)] \\ = \mathbb{E}[\xi^T(t_k)\Lambda(k)u_k + u_k^T\Lambda^T(k)\xi(t_k) + u_k^T\Pi(k)u_k] \\ + \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \mathbf{Tr}[\hat{X}(t)\hat{B}_v(t)V\hat{B}_v^T(t)]dt\right].$$

Let $\eta_1(k) = [I_m - \Pi(k)^\dagger\Pi(k)]\Psi(k)\xi(t_k)$, $\eta_2(k) = [I_m - \Pi(k)^\dagger\Pi(k)]\varphi(k)$. Based on (15) we obtain that

$$\Lambda(k)\eta_i(k) = 0, \quad i = 1,\ 2. \tag{18}$$

On the other hand, from $\Pi(k)\Pi(k)^\dagger\Pi(k) = \Pi(k)$, we have

$$\Pi(k)\eta_i(k) = 0, \quad i = 1,\ 2, \quad 0 \le k \le N-1. \tag{19}$$

Now, employing (18), (19) together with the second align of (13) we obtain:

$$\mathbb{E}\left[\int_{t_k}^{t_{k+1}} \xi^T(t)\hat{M}(t)\xi(t)dt\right] + \mathbb{E}[u_k^T\hat{R}(k)u_k] \\ + \mathbb{E}[\xi^T(t_{k+1})\hat{X}(t_{k+1}^-)\xi(t_{k+1})] - \mathbb{E}[\xi^T(t_k)\hat{X}(t_k^-)\xi(t_k)] \\ = \mathbb{E}[(u_k - u_{\psi\varphi k})^T\Pi(k)(u_k - u_{\psi\varphi k})] \\ + \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \mathbf{Tr}[\hat{X}(t)\hat{B}_v(t)V\hat{B}_v^T(t)]dt\right].$$

Summing for $k = 0$ up to $k = N-1$ we have the equality

$$\mathcal{J}_d(\mathbf{u}, \xi_0) = \mathcal{J}_d(\mathbf{u}_{\psi\varphi}, \xi_0) + \sum_{k=0}^{N-1} \mathbb{E}[(u_k - u_{\psi\varphi k})^T \\ \times \Pi(k)(u_k - u_{\psi\varphi k})] \tag{20}$$

for all $\mathbf{u} \in \mathcal{U}_{ad}^d$, $\mathcal{J}_d(\mathbf{u}_{\psi\varphi}, \xi_0)$ being introduced via (17). Invoking (14), we deduce from (20) that $\mathcal{J}_d(\mathbf{u}, \xi_0) \ge \mathcal{J}_d(\mathbf{u}_{\psi\varphi}, \xi_0)$ for all $\mathbf{u} \in \mathcal{U}_{ad}^d$ which confirms the optimality of the control defined in (16) and that the minimal value of cost (12) is provided by (17). Thus the proof ends. □

Remark 5 *If $\hat{R}(k) + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k) > 0$ for $0 \leq k \leq N-1$, then $\Pi(k)^\dagger$ coincides with $\Pi(k)^{-1}$ and in this case the control (16) reduces to*

$$\tilde{u}_k = \tilde{\mathbf{F}}(k)\tilde{\xi}(t_k) \tag{21}$$

where $0 \leq k \leq N-1$,

$$\tilde{\mathbf{F}}(k) = -[\hat{R}(k) + \hat{B}_d^T(k)\hat{X}(t_k)\hat{B}_d(k)]^{-1}\hat{B}_d^T(k)\hat{X}(t_k)\hat{A}_d(k). \tag{22}$$

Furthermore, in this case the LQ optimization problem described by the performance criterion (12) and the class of admissible controls $\mathcal{U}_{ad}^d$ has a unique optimal control and it is given by (21) and (22).

4 Solution of LQ optimal control with piecewise constant controls

As we have shown, the equality (10) enables us to obtain the piecewise constant control $\tilde{u}(t)$ which minimize the cost (2). To this end we have to search the control $\tilde{\mathbf{u}} \in \mathcal{U}_{ad}^d$ that minimizes the cost (9).

We shall apply Theorem 4 in the special case when $\hat{A}_k(t) = \mathcal{A}_k(t), k = 1, 2, \hat{B}_v(t) = \mathcal{B}_v(t), \hat{A}_d(k) = \mathcal{A}_d, \hat{B}_d(k) = \mathcal{B}_d$ described in (8) and $\hat{G} = \mathcal{G}, \hat{M}(t) = \mathcal{M}(t), \hat{R}(k) = \mathcal{R}(k)$ defined in (9).

4.1 The case of weights matrices which satisfy the assumption $\mathbf{H_3}$ a)

With the notations used for the controlled system (7) and the performance criterion (9) we rewrite the differential align with finite jumps of Riccati type (13) in the form:

$$\begin{aligned} -\dot{X}(t) &= \mathcal{A}_0^T(t)X(t) + X(t)\mathcal{A}_0(t) \\ &\quad + \mathcal{A}_1^T(t)X(t)\mathcal{A}_1(t) + \mathcal{M}(t), \quad t_k \leq t < t_{k+1}, \end{aligned} \tag{23a}$$

$$\begin{aligned} X(t_k^-) &= \mathcal{A}_d^T X(t_k)\mathcal{A}_d - \mathcal{A}_d^T X(t_k)\mathcal{B}_d \\ &\quad \times \left(\mathcal{R}(k) + \mathcal{B}_d^T X(t_k)\mathcal{B}_d\right)^\dagger \mathcal{B}_d^T X(t_k)\mathcal{A}_d, \\ k &= 0,\ 1, \ldots N-1. \end{aligned} \tag{23b}$$

First, we show that in this case the assumption H_4) specialized for the differential align with finite jumps of Riccati type (23) is fulfilled. To this end, we prove:

Proposition 6 *Assume that the assumption $\mathbf{H_2}$) and $\mathbf{H_3}$) a) hold. Under these conditions the differential align with finite jumps (23) has a unique solution $\tilde{X}(\cdot) : [0, \tau] \to \mathcal{S}_p$ satisfying the terminal condition $\tilde{X}(\tau) = \mathcal{G}$. This solution has the properties:*

$$\begin{aligned} &\text{(i) } \tilde{X}(t) \geq 0,\ t \in [0, \tau], \\ &\text{(ii) } \mathcal{R}(k) + \mathcal{B}_d^T X(t_k)\mathcal{B}_d > 0,\ k = 0,\ 1, \ldots, N-1. \end{aligned} \tag{24}$$

Proof: Let $\mathcal{L}(t) : \mathcal{S}_p \to \mathcal{S}_p$ be the linear operator defined by

$$\mathcal{L}(t)[Y] = \mathcal{A}_0(t)Y + Y\mathcal{A}_0^T(t) + \mathcal{A}_1(t)Y\mathcal{A}_1^T(t) \tag{25}$$

for all $Y \in \mathcal{S}_p$. Let $\mathbb{T}(t,s)$ be the linear evolution operator on $\mathcal{S}_p$, defined by the linear differential align

$$\dot{Y}(t) = \mathcal{L}(t)[Y(t)]. \tag{26}$$

According to Example 3.10 (b) from [10] we deduce that $\mathbb{T}(t,s)$ and its adjoint operator $\mathbb{T}^*(t,s)$ are positive operators, that is $\mathbb{T}(t,s)[Y] \geq 0$ and $\mathbb{T}^*(t,s)[Y] \geq 0$, if $Y \geq 0$ for all $t \geq s \geq 0$.

For each $t \in [t_{N-1}, \tau]$ the solution $\tilde{X}(\cdot)$ of the differential align (23) has the representation:

$$\tilde{X}(t) = \mathbb{T}^*(\tau,t)[\mathcal{G}] + \int_t^\tau \mathbb{T}^*(s,t)[\mathcal{M}(s)]ds. \tag{27}$$

Invoking the assumption $\mathbf{H_3}$ a) together with positivity of the linear operator $\mathbb{T}^*(s,t)[\cdot]$ for $s \geq t$ we may conclude that $\tilde{X}(t) \geq 0, t_{N-1} \leq t \leq \tau$. Hence, $\mathcal{R}(N-1) + \mathcal{B}_d^T\tilde{X}(t_{N-1})\mathcal{B}_d > 0$ because $\mathcal{R}(N-1) > 0$. We set $\tilde{\mathbb{F}}(N-1) = -(\mathcal{R}(N-1) + \mathcal{B}_d^T\tilde{X}(t_{N-1})\mathcal{B}_d)^{-1}\mathcal{B}_d^T\tilde{X}(t_{N-1})\mathcal{A}_d$. By direct calculations one obtains from (23) that

$$\begin{aligned}\tilde{X}(t_{N-1}^-) &= (\mathcal{A}_d + \mathcal{B}_d\tilde{\mathbb{F}}(N-1))^T\tilde{X}(t_{N-1})(\mathcal{A}_d + \mathcal{B}_d\tilde{\mathbb{F}}(N-1)) \\ &\quad + \tilde{\mathbb{F}}^T(N-1)\mathcal{R}(N-1)\tilde{\mathbb{F}}(N-1).\end{aligned}$$

Thus, we deduce that $\tilde{X}(t_{N-1}^-) \geq 0$.

For $t \in [t_{N-2}, t_{N-1}]$ we have

$$\tilde{X}(t) = \mathbb{T}^*(t_{N-1},t)[\tilde{X}(t_{N-1}^-)] + \int_t^{t_{N-1}} \mathbb{T}^*(s,t)[\mathcal{M}(s)]ds. \tag{28}$$

This allows us to conclude that $\tilde{X}(t) \geq 0$, for all $t_{N-2} \leq t \leq t_{N-1}$.

Let us assume by induction that $\tilde{X}(t) \geq 0$ for $t_k \leq t \leq t_{k+1}$. Since $\mathcal{R}(k) > 0$ we deduce that

$$\mathcal{R}(k) + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d > 0. \tag{29}$$

Thus we may deduce

$$\tilde{\mathbb{F}}(k) = -\left(\mathcal{R}(k) + \mathcal{B}_d^T\tilde{X}(t_k)\mathcal{B}_d\right)^{-1}\mathcal{B}_d^T\tilde{X}(t_k)\mathcal{A}_d. \tag{30}$$

By direct calculation we obtain from (23) and (30) that $\tilde{X}(t_k^-) = (\mathcal{A}_d + \mathcal{B}_d\tilde{\mathbb{F}}(k))^T\tilde{X}(t_k)(\mathcal{A}_d + \mathcal{B}_d\tilde{\mathbb{F}}(k)) + \tilde{\mathbb{F}}^T(k)\mathcal{R}(k)\tilde{\mathbb{F}}(k)$. This shows that $\tilde{X}(t_k^-) \geq 0$. Furthermore, for $t_{k-1} \leq t \leq t_k$ we have the representation formula:

$$\tilde{X}(t) = \mathbb{T}^*(t_k,t)[\tilde{X}(t_k^-)] + \int_t^{t_k} \mathbb{T}^*(s,t)[\mathcal{M}(s)]ds. \tag{31}$$

This leads to $\tilde{X}(t) \geq 0$ for all $t_{k-1} \leq t \leq t_k$. In this way we obtain that (29) holds for t_k replaced by t_{k-1}. Thus, we have shown by induction that (29) holds and $\tilde{\mathbb{F}}(k)$ can be computed via (30) for any $0 \leq k \leq N-1$. Thus the proof is complete. □

Let $\begin{pmatrix} \tilde{X}_{11}(t) & \tilde{X}_{12}(t) \\ \tilde{X}_{12}^T(t) & \tilde{X}_{22}(t) \end{pmatrix}$ be the partition of the unique solution of the differential align (23) satisfying $\tilde{X}(\tau) = \mathcal{G}$, where $\tilde{X}_{11}(t) \in \mathbb{R}^{n\times n}$, $\tilde{X}_{22}(t) \in \mathbb{R}^{m\times m}$ and $\tilde{X}_{12}(t) \in \mathbb{R}^{n\times m}$. Using the structure given in (8) and (9) for the coefficients of (23) we obtain the following partition of this align:

$$
\begin{aligned}
-\frac{d}{dt}\tilde{X}_{11}(t) &= A_0^T(t)\tilde{X}_{11}(t) + \tilde{X}_{11}(t)A_0(t) \\
&\quad + A_1^T(t)\tilde{X}_{11}(t)A_1(t) + M(t) && \text{(32a)} \\
-\frac{d}{dt}\tilde{X}_{12}(t) &= A_0^T(t)\tilde{X}_{12}(t) + \tilde{X}_{11}(t)B_0(t) \\
&\quad + A_1^T(t)\tilde{X}_{11}(t)B_1(t) && \text{(32b)} \\
-\frac{d}{dt}\tilde{X}_{22}(t) &= B_0^T(t)\tilde{X}_{12}(t) + \tilde{X}_{12}^T(t)B_0(t) \\
&\quad + B_1^T(t)\tilde{X}_{11}(t)B_1(t),\ t_k \leq t < t_{k+1}, && \text{(32c)} \\
\tilde{X}_{11}(t_k^-) &= \tilde{X}_{11}(t_k) - \tilde{X}_{12}(t_k) \\
&\quad \times \left(\int_{t_k}^{t_{k+1}} R(s)ds + \tilde{X}_{22}(t_k)\right)^{-1} \tilde{X}_{12}^T(t_k) && \text{(32d)} \\
\tilde{X}_{12}(t_k^-) &= 0,\ \tilde{X}_{22}(t_k^-) = 0, \quad k = 0,\ 1,\ 2, \ldots, N-1, \\
\tilde{X}_{11}(\tau) &= G,\ \tilde{X}_{12}(\tau) = 0, \tilde{X}_{22}(\tau) = 0.
\end{aligned}
$$

Specializing the result derived in Theorem 4 to the optimal control problem discussed by the cost functional (9) and the class of admissible controls $\mathcal{U}_{ad}^d$ we obtain via Remark 3.1 and equality (10) the following result:

Theorem 7 *Assume that the assumptions* $\mathbf{H_1}$, $\mathbf{H_2}$ *and* $\mathbf{H_3}$ *a) are fulfilled. Under these conditions, there exists a unique piecewise constant optimal control* $\tilde{u} \in \mathcal{U}_{ad}$ *which minimizes the cost functional (2) over* $\mathcal{U}_{ad}$. *This control is described by*

$$\tilde{u}(t) = \tilde{F}(k)\tilde{x}(t_k), \quad t_k \leq t < t_{k+1},\ k = 0,\ 1, \ldots, N-1, \tag{33}$$

where

$$\tilde{F}(k) = -\left(\int_{t_k}^{t_{k+1}} R(s)ds + \tilde{X}_{22}(t_k)\right)^{-1} \tilde{X}_{12}^T(t_k) \tag{34}$$

and $\tilde{x}(t_k)$ *are the measured values of the solution* $\tilde{x}(\cdot)$ *of the problem with given initial values:*

$$
\begin{aligned}
d\tilde{x}(t) = \Big(A_0(t)\tilde{x}(t) + B_0(t)\tilde{F}(k)\tilde{x}(t_k)\Big)dt + \Big(A_1(t)\tilde{x}(t) \\
+ B_1(t)\tilde{F}(k)\tilde{x}(t_k)\Big)dw(t) + B_v(t)dv(t),
\end{aligned} \tag{35}
$$

$t_k \leq t < t_{k+1}, \tilde{x}(0) = x_0$. *The minimal value of the cost functional (2) is:*

$$J(\tilde{u}, x_0) = x_0^T \tilde{X}_{11}(0^-)x_0 + \int_0^\tau \mathbf{Tr}[\tilde{X}_{11}(t)B_v(t)VB_v^T(t)]dt. \tag{36}$$

Proof: According to Remark 5, (24) allows us to infer that there exists a unique control that minimizes the cost functional (9) over the family of the admissible controls $\mathcal{U}_{ad}^d$. In this special case, the gain matrix (22) of the optimal control is $\tilde{\mathbb{F}}(k)$ introduced via (30). Using (8) and (9) we obtain that $\tilde{\mathbb{F}}(k) = (\tilde{F}(k) \quad 0)$, where $\tilde{F}(k)$ is defined in (34). The analogous of the optimal control (21) is:

$$\tilde{u}_k = \tilde{F}(k)\tilde{x}(t_k), \tag{37}$$

where $\tilde{F}(k)$ is defined in (34) and $\tilde{x}(t_k)$ are the values of the first n components of the solution $\xi(t) = (x^T(t) \quad u^T(t))^T$ of the problem with given initial values

$$d\xi(t) = \mathcal{A}_0(t)\xi(t)dt + \mathcal{A}_1(t)\xi(t)dw(t), \ t_k < t \leq t_{k+1}, \tag{38a}$$

$$\xi(t_k^+) = \Big(\mathcal{A}_d + \mathcal{B}_d \begin{pmatrix} \tilde{F}(k) & 0 \end{pmatrix}\Big)\xi(t_k), \tag{38b}$$

where $k = 0, \ 1, \ldots, N-1$, $\xi(0) = \begin{pmatrix} x_0^T & (\tilde{F}(0)x_0)^T \end{pmatrix}^T$.

By direct calculation involving (8) shows that the solution of the problem with initial given value (38) is $\tilde{\xi}(t) = (\tilde{x}^T(t) \qquad \tilde{u}^T(t))^T$, where $\tilde{x}(\cdot)$ is just the solution of the problem with given initial values (35) while $\tilde{u}(\cdot)$ is the control (33). The minimal value $J(\tilde{u}, x_0)$ of the cost functional (2) is obtained directly from (10) and (17) taking into account that (32) leads to $\tilde{X}_{12}(0^-) = 0, \tilde{X}_{22}(0^-) = 0$. Thus the proof is complete. □

In conclusion, in order to design the optimal piecewise constant control that minimizes cost (2) under the assumption H_3) a), one may proceed as follows:

Step 1. One integrates the system of backward differential align (32) on the interval $[0, \tau]$.

Step 2. One computes the feedback gain matrices $\tilde{F}(k)$ as in (34). The desired optimal control is given by (33)

4.2 The case of weights matrices that satisfy the assumption H_3) b)

In this case, (9) yields $\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(p)\times(p)}$, $\mathcal{G} \geq 0$, $\mathcal{M} \equiv 0, \mathcal{M} \in \mathbb{R}^{(p)\times(p)}, \mathcal{R}(k) \equiv 0, \mathcal{R}(k) \in \mathbb{R}^{(m)\times(m)}, 0 \leq k \leq N-1$. The differential align with finite jumps of Riccati type (23) becomes:

$$\begin{aligned} -\dot{X}(t) &= \mathcal{A}_0^T(t)X(t) + X(t)\mathcal{A}_0(t) \\ &\quad + \mathcal{A}_1^T(t)X(t)\mathcal{A}_1(t) \quad t_k \leq t < t_{k+1}, \end{aligned} \tag{39a}$$

$$\begin{aligned} X(t_k^-) &= \mathcal{A}_d^T X(t_k)\mathcal{A}_d(t) - \mathcal{A}_d^T X(t_k)\mathcal{B}_d \left(\mathcal{B}_d^T X(t_k)\mathcal{B}_d\right)^\dagger \\ &\quad \times \mathcal{B}_d^T X(t_k)\mathcal{A}_d, \ k = 0, \ 1, \ldots, N-1, \end{aligned} \tag{39b}$$

where $\mathcal{A}_k(t), k = 1, 2, \mathcal{A}_d, \mathcal{B}_d$ are those introduced in (8). The next result shows that the assumption H_4) is fulfilled in the special case of the differential align with finite jumps of Riccati type (39).

Proposition 8 *Assume that the assumptions* $\mathbf{H_2}$ *and* $\mathbf{H_3}$ *b) are fulfilled. Under these conditions, the differential align with finite jumps of Riccati type (39) has a unique solution* $\tilde{\tilde{X}} : [0, \tau] \to \mathcal{S}_p$ *satisfying the terminal condition* $\tilde{X}(\tau) = \mathcal{G}$*. This solution has the properties:*

(i) $\tilde{\tilde{X}}(t) \geq 0$ *for all* $t \in [0, \tau]$*;*

$$\begin{aligned}&\text{(ii)}[I_m - \mathcal{B}_d^T \tilde{\tilde{X}}(t_k)\mathcal{B}_d \left(\mathcal{B}_d^T \tilde{\tilde{X}}(t_k)\mathcal{B}_d\right)^{\dagger}]\mathcal{B}_d^T \tilde{\tilde{X}}(t_k)\mathcal{A}_d = 0 \\ &k = 0,\ 1, \ldots, N-1.\end{aligned} \tag{40}$$

To proof this result we need a generalized version of the Schur Lemma.

Lemma 9 *(generalized Schur Lemma [1, 9].) Let* $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in \mathcal{S}_{\hat{n}_1+\hat{n}_2}$ *such that* $U_{jj} \in \mathcal{S}_{\hat{n}_j}, j = 1, 2$*. The following are equivalent:*

(i) $U \geq 0$*,*

(ii) $U_{22} \geq 0$, $U_{11} - U_{12}U_{22}^{\dagger}U_{12}^T \geq 0$, $U_{12} = U_{12}U_{22}^{\dagger}U_{22}$ *.*

Proof: Employing the linear evolution operator $\mathbb{T}(s, t)$ defined by the linear differential align (26) on the linear space $\mathcal{S}_p$, we obtain the following representation of the solution $\tilde{\tilde{X}}(t)$ for $t \in [t_{N-1}, \tau]$

$$\tilde{\tilde{X}}(t) = \mathbb{T}^*(\tau, t)[\mathcal{G}]. \tag{41}$$

Hence, $\tilde{\tilde{X}}(t) \geq 0, t \in [t_{N-1}, \tau]$ because $\mathbb{T}^*(\tau, t)[\cdot]$ is a positive operator and $\mathcal{G} \geq 0$.

Let $\tilde{\tilde{X}}(t) = \begin{pmatrix} \tilde{\tilde{X}}_{11}(t) & \tilde{\tilde{X}}_{12}(t) \\ \tilde{\tilde{X}}_{12}^T(t) & \tilde{\tilde{X}}_{22}(t) \end{pmatrix}$ be such that $\tilde{\tilde{X}}_{11}(t) \in \mathcal{S}_n$ and $\tilde{\tilde{X}}_{22}(t) \in \mathcal{S}_m$. Using the implication (i) $\to$ (ii) in Lemma 9 in the case of the matrix $U = \tilde{\tilde{X}}(t_{N-1}) \geq 0$ we obtain that

$$\begin{aligned}&\tilde{\tilde{X}}_{22}(t_{N-1}) \geq 0, \\ &\tilde{\tilde{X}}_{11}(t_{N-1}) - \tilde{\tilde{X}}_{12}(t_{N-1})\tilde{\tilde{X}}_{22}(t_{N-1})\tilde{\tilde{X}}_{12}^T(t_{N-1}) \geq 0,\end{aligned} \tag{42}$$

and $[I_m - \tilde{\tilde{X}}_{22}(t_{N-1})\tilde{\tilde{X}}_{22}^{\dagger}(t_{N-1})]\tilde{\tilde{X}}_{12}^T(t_{N-1}) = 0$.
Using the structure of the matrices $\mathcal{A}_d, \mathcal{B}_d$ given by (8) we deduce that (42) coincides with

$$\begin{aligned}&\mathcal{A}_d^T \tilde{\tilde{X}}(t_{N-1})\mathcal{A}_d - \mathcal{A}_d^T \tilde{\tilde{X}}(t_{N-1})\mathcal{B}_d \\ &\quad \times \left(\mathcal{B}_d^T \tilde{\tilde{X}}(t_{N-1})\mathcal{B}_d\right)^{\dagger} \mathcal{B}_d^T \tilde{\tilde{X}}(t_{N-1})\mathcal{A}_d \geq 0.\end{aligned} \tag{43}$$

From (39) for $k = N-1$ we obtain via (43) that $\tilde{\tilde{X}}(t_{N-1}^-) \geq 0$. Further on, the representation $\tilde{\tilde{X}}(t) = \mathbb{T}^*(t_{N-1}, t)[\tilde{\tilde{X}}(t_{N-1}^-)]$ yields $\tilde{\tilde{X}}(t) \geq 0$ for all $t_{N-2} \leq t \leq t_{N-1}$.

Let us assume that for some $0 \leq k \leq N-1$:

$$\tilde{\tilde{X}}(t) \geq 0, \quad t_k \leq t \leq t_{k+1}. \tag{44}$$

Let

$$\tilde{\tilde{X}}(t) = \begin{pmatrix} \tilde{\tilde{X}}_{11}(t) & \tilde{\tilde{X}}_{12}(t) \\ \tilde{\tilde{X}}_{12}^T(t) & \tilde{\tilde{X}}_{22}(t) \end{pmatrix} \tag{45}$$

be the partition of the solution $\tilde{\tilde{X}}(\cdot)$ such that $\tilde{\tilde{X}}_{11}(t) \in \mathcal{S}_n$ and $\tilde{\tilde{X}}_{22}(t) \in \mathcal{S}_m$. Applying (i) $\rightarrow$ (ii) from Lemma 9 in the case of the matrix $U = \tilde{\tilde{X}}(t_k) \geq 0$ we obtain:

$$\tilde{\tilde{X}}_{22}(t_k) \geq 0,\ \tilde{\tilde{X}}_{11}(t_k) - \tilde{\tilde{X}}_{12}(t_k)\tilde{\tilde{X}}_{22}^{\dagger}(t_k)\tilde{\tilde{X}}_{12}^T(t_k) \geq 0, \tag{46}$$

$$[I_m - \tilde{\tilde{X}}_{22}(t_k)\tilde{\tilde{X}}_{22}^{\dagger}(t_k)]\tilde{\tilde{X}}_{12}^T(t_k) = 0. \tag{47}$$

Using again the structure of the matrices $\mathcal{A}_d, \mathcal{B}_d$ given in (8) we obtain that (45) coincides with $\tilde{\tilde{X}}(t_k) \geq 0$, $\mathcal{A}_d^T\tilde{\tilde{X}}(t_k)\mathcal{A}_d - \mathcal{A}_d^T\tilde{\tilde{X}}(t_k)\mathcal{B}_d\left(\mathcal{B}_d^T\tilde{\tilde{X}}(t_k)\mathcal{B}_d\right)^{\dagger}\mathcal{B}_d^T\tilde{\tilde{X}}(t_k)\mathcal{A}_d \geq 0$. The second align from (39) yields $\tilde{\tilde{X}}(t_k^-) \geq 0$.

Furthermore, the representation $\tilde{\tilde{X}}(t_k) = \mathbb{T}^*(t_k, t)[\tilde{\tilde{X}}(t_k^-)]$ allows us to infer that $\tilde{\tilde{X}}(t) \geq 0$ for all $t_{k-1} \leq t \leq t_k$. In this way one obtains inductively that (44) holds for any $0 \leq k \leq N-1$, which confirms the validity of (i) from the statement. Finally, we remark that (40) coincides with (47), this ends the proof. □

According to (45) and (8) we obtain the following partition of (39):

$$\begin{aligned} -\frac{d}{dt}\tilde{\tilde{X}}_{11}(t) &= A_0^T(t)\tilde{\tilde{X}}_{11}(t) + \tilde{\tilde{X}}_{11}(t)A_0(t) \\ &\quad + A_1^T(t)\tilde{\tilde{X}}_{11}(t)A_1(t), \end{aligned} \tag{48a}$$

$$\begin{aligned} -\frac{d}{dt}\tilde{\tilde{X}}_{12}(t) &= A_0^T(t)\tilde{\tilde{X}}_{12}(t) + \tilde{\tilde{X}}_{11}(t)B_0(t) \\ &\quad + A_1^T(t)\tilde{\tilde{X}}_{11}(t)B_1(t), \end{aligned} \tag{48b}$$

$$\begin{aligned} -\frac{d}{dt}\tilde{\tilde{X}}_{22}(t) &= B_0^T(t)\tilde{\tilde{X}}_{12}(t) + \tilde{\tilde{X}}_{12}^T(t)B_0(t) \\ &\quad + B_1^T(t)\tilde{\tilde{X}}_{11}(t)B_1(t),\ t_k \leq t \leq t_{k+1}, \end{aligned} \tag{48c}$$

$$\begin{aligned} \tilde{\tilde{X}}_{11}(t_k^-) &= \tilde{\tilde{X}}_{11}(t_k) - \tilde{\tilde{X}}_{12}(t_k)\left(\tilde{\tilde{X}}_{22}(t_k)\right)^{\dagger}\tilde{\tilde{X}}_{12}^T(t_k) \\ \tilde{\tilde{X}}_{12}(t_k^-) &= 0,\ \tilde{\tilde{X}}_{22}(t_k^-) = 0,\ k = 0,\ 1,\ 2, \ldots, N-1, \\ \tilde{\tilde{X}}_{11}(\tau) &= G,\ \tilde{\tilde{X}}_{12}(\tau) = 0,\ \tilde{\tilde{X}}_{22}(\tau) = 0. \end{aligned} \tag{48d}$$

The main result of this subsection is:

Theorem 10 *Assume that the assumptions* $\mathbf{H_1}$, $\mathbf{H_2}$ *and* $\mathbf{H_3}$ *b) are fulfilled. Under these conditions for arbitrary* $\Psi(k) \in \mathbb{R}^{m\times m}, \varphi(k) \in \mathbb{R}^m$, *the piecewise constant controls*

$$\begin{aligned} u_{\psi\varphi}(t) &= -[\tilde{\tilde{X}}_{22}^{\dagger}(t_k)\tilde{\tilde{X}}_{12}^T(t_k) + (I_m - \tilde{\tilde{X}}_{22}^{\dagger}(t_k)\tilde{\tilde{X}}_{22}(t_k)\Psi(k)]\tilde{\tilde{x}}(t_k) \\ &\quad - (I_m - \tilde{\tilde{X}}_{22}^{\dagger}(t_k)\tilde{\tilde{X}}_{22}(t_k))\varphi(k),\ t_k \leq t < t_{k+1}, \end{aligned} \tag{49}$$

$k = 0, 1, \ldots, N-1$, *minimizes the value of the functional (5) over the class of admissible controls* $\mathcal{U}_{ad}$. $\tilde{\tilde{x}}(t_k)$ *are the values of the solution* $\tilde{\tilde{x}}(t)$ *of the system obtained when the control (49) are plugged in the system (1). The minimal value of the performance criterion (5) is*

$$J(u_{\Psi\varphi}, x_0) = x_0^T \tilde{\tilde{X}}_{11}(0^-)x_0 + \int_0^\tau \mathbf{Tr}[\tilde{\tilde{X}}_{11}(t)B_v(t)VB_v^T(t)]dt. \tag{50}$$

Proof: The value at $t = t_k$ of the (49) coincides with the special form of control (16) in the case of the optimal control problem described by the system with finite jumps:

$$d\xi(t) = \mathcal{A}_0(t)\xi(t)dt + \mathcal{A}_1(t)\xi(t)dw(t), \ t_k < t \le t_{k+1} \tag{51a}$$

$$\xi(t_k^+) = \mathcal{A}_d\xi(t_k) + \mathcal{B}_d u_k \tag{51b}$$

and the cost functional $\mathcal{J}(u, \xi_0) = \mathbb{E}[\xi^T(\tau)\mathcal{G}\xi(\tau)]$. Applying Theorem 4 in the case of the linear quadratic optimal control problem described by the functional (52) one obtains the desired conclusion. □

4.3 Some procedural issues

In this subsection we provide some procedures for numerical computation of the values of the matrices $\tilde{X}(t_k)$ and $\tilde{\tilde{X}}(t_k)$ respectively, involved in the computation of the gain matrices of the optimal controls (33)-(34) and (49). The methods discussed in this subsection can be viewed as alternative methods to the ones based on the direct integration of the systems of matrix differential aligns with finite jumps (32) and (48), respectively.

From the first align of (23) we obtain:

$$\tilde{X}(t_k) = \mathbb{T}^*(t_{k+1}, t_k)[X(t_{k+1}^-)] + \int_{t_k}^{t_{k+1}} \mathbb{T}^*(s, t_k)[\mathcal{M}(s)]ds \tag{52}$$

and from the first align of (39) we obtain

$$\tilde{\tilde{X}}(t_k) = \mathbb{T}^*(t_{k+1}, t_k)[\tilde{\tilde{X}}(t_{k+1}^-)] \tag{53}$$

$\mathbb{T}^*(s,t)$ being the adjoint operator of the linear evolution operator defined by the linear differential align (26) on the linear space $\mathcal{S}_p$. Plugging (52) in the second align of (23), and (53) in the second align of (39), respectively, we obtain the following backward discrete-time aligns for the computation of $\tilde{X}(t_k^-)$ and $\tilde{\tilde{X}}(t_k^-)$ respectively:

$$\begin{aligned}\tilde{X}(t_k^-) = {} & \Pi_1(k)[\tilde{X}(t_{k+1}^-)] - \big(\Pi_2(k)[\tilde{X}(t_{k+1}^-)] \\ & + \mathbb{L}(k)\big)\big(\Pi_3(k)[\tilde{X}(t_{k+1}^-)] + \mathbb{R}(k)\big)^{-1} \\ & \times (\Pi_2(k)[\tilde{X}(t_{k+1}^-)] + \mathbb{L}(k))^T + \mathbb{M}(k),\end{aligned} \tag{54a}$$

$$\begin{aligned}\tilde{\tilde{X}}(t_k^-) = {} & \Pi_1(k)[\tilde{\tilde{X}}(t_k^-)] - \Pi_2(k)[\tilde{\tilde{X}}(t_k^-)] \\ & \times (\Pi_3(k)[\tilde{\tilde{X}}(t_k^-)])^\dagger(\Pi_2(k)[\tilde{\tilde{X}}(t_k^-)])^T,\end{aligned} \tag{54b}$$

$k = N-1, ..., 0$ with $\tilde{X}(t_N^-) = \tilde{\tilde{X}}(t_N^-) = \mathcal{G}$, where $X \to \Pi_1(k)[X] : \mathcal{S}_p \to \mathcal{S}_p$, $X \to \Pi_2(k)[X] : \mathcal{S}_p \to \mathbb{R}^{(p)\times m}$, $X \to \Pi_3(k)[X] : \mathcal{S}_p \to \mathcal{S}_m$ are defined by

$$\Pi_1(k)[X] = \mathcal{A}_d^T \mathbb{T}^*(t_{k+1}, t_k)[X]\mathcal{A}_d, \tag{55a}$$

$$\Pi_2(k)[X] = \mathcal{A}_d^T \mathbb{T}^*(t_{k+1}, t_k)[X]\mathcal{B}_d, \tag{55b}$$

$$\Pi_3(k)[X] = \mathcal{B}_d^T \mathbb{T}^*(t_{k+1}, t_k)[X]\mathcal{B}_d. \tag{55c}$$

In (54a) we have used also the notations:

$$\mathbb{M}(k) = \mathcal{A}_d^T \int_{t_k}^{t_{k+1}} \mathbb{T}^*(s, t_k)[\mathcal{M}(s)]ds\mathcal{A}_d, \tag{56a}$$

$$\mathbb{L}(k) = \mathcal{A}_d^T \int_{t_k}^{t_{k+1}} \mathbb{T}^*(s, t_k)[\mathcal{M}(s)]ds\mathcal{B}_d, \tag{56b}$$

$$\mathbb{R}(k) = \int_{t_k}^{t_{k+1}} R(s)ds + \mathcal{B}_d^T \int_{t_k}^{t_{k+1}} \mathbb{T}^*(s, t_k)[\mathcal{M}(s)]ds\mathcal{B}_d. \tag{56c}$$

The matrices $\tilde{X}(t_k^-)$ and $\tilde{\tilde{X}}(t_k^-)$, $k = 1, 2, ..., N$ obtained via (54a) and (54b), respectively, are introduced in (52) or (53), respectively, to obtain the matrices $\tilde{X}(t_k)$, or $\tilde{\tilde{X}}(t_k)$, respectively, involved in (34) and (49). Further we shall analyse the case when both the system (1) and the cost functional (2) are time invariant and $t_{k+1} - t_k = h > 0$ for all $k = 0, 1, ..., N-1$. This means that $A_k(t) = A_k$, $B_k(t) = B_k$, $k = 0, 1$, $M(t) = M$, $R(t) = R$ for all $t \in [0, \tau]$. In this case, we have $\mathbb{T}^*(t, s) = e^{\mathcal{L}^*(t-s)}$, which leads to

$$T^*(t_{k+1}, t_k) = e^{\mathcal{L}^* h} = \sum_{j=0}^{\infty} \frac{h^j}{j!} \mathcal{L}^{*j} \tag{57}$$

where $\mathcal{L}^* : \mathcal{S}_p \to \mathcal{S}_p$ is described by

$$\mathcal{L}^*[X] = \mathcal{A}_0^T X + X\mathcal{A}_0 + \mathcal{A}_1^T X \mathcal{A}_1 \tag{58}$$

for all $X \in \mathcal{S}_p$, $\mathcal{A}_k$, $k = 0, 1$ being introduced in (8).

For the numerical computation of $e^{\mathcal{L}^* h}$ one may use the truncation of the series from (57), i.e.

$$e^{\mathcal{L}^* h}[X] \simeq \sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[X] \tag{59}$$

where $p \geq 1$ is a sufficiently large natural number, such that

$$\frac{h^{p+1}}{(p+1)!} \lambda_{max}\{(\mathcal{L}^{(p+1)})^*[X]\} < \text{toll}. \tag{60}$$

Also in this special case we have

$$\int_{t_k}^{t_{k+1}} e^{\mathcal{L}^*(s-t_k)}[\mathcal{M}]ds = \int_0^h e^{\mathcal{L}s}[\mathcal{M}]ds$$
$$\simeq \sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!} \mathcal{L}^{*j}[\mathcal{M}], \tag{61}$$

where $p \geq 1$ is such that (60) is satisfied for $X = \mathcal{M}$. Thus we obtained the following approximations of the operators defined in (55) and of the matrices introduced by (56)

$$\Pi_1(k)[X] \simeq \mathcal{A}_d^T \left(\sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[X] \right) \mathcal{A}_d, \tag{62a}$$

$$\Pi_2(k)[X] \simeq \mathcal{A}_d^T \left(\sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[X] \right) \mathcal{B}_d, \tag{62b}$$

$$\Pi_3(k)[X] \simeq \mathcal{B}_d^T \left(\sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[X] \right) \mathcal{B}_d \tag{62c}$$

and

$$\mathbb{M}(k) \simeq \mathcal{A}_d^T \left(\sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!} \mathcal{L}^{*j}[\mathcal{M}] \right) \mathcal{A}_d, \tag{63a}$$

$$\mathbb{L}(k) \simeq \mathcal{A}_d^T \left(\sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!} \mathcal{L}^{*j}[\mathcal{M}] \right) \mathcal{B}_d, \tag{63b}$$

$$\mathbb{R}(k) \simeq hR + \mathcal{B}_d^T \left(\sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!} \mathcal{L}^{*j}[\mathcal{M}] \right) \mathcal{B}_d. \tag{63c}$$

As it was expected both the operators from (62) and the matrices from (63) are not depending of k. The iterations $\mathcal{L}^{*j}$ of the operator $\mathcal{L}^*$ introduced by (58) involved in (62) and (63) are obtained as follows

$$\begin{aligned} \mathcal{L}^{*(j)}[X] &= \mathcal{A}_0^T \mathcal{L}^{*(j-1)}[X] + \mathcal{L}^{*(j-1)}[X]\mathcal{A}_0 \\ &\quad + \mathcal{A}_1^T \mathcal{L}^{*(j-1)}[X]\mathcal{A}_1, \ \mathcal{L}^{*(0)}[X] = X. \end{aligned} \tag{64}$$

Summing up, the following algorithm may be released:

Step 0. Taking $X = \mathcal{M}$ (introduced in (9)) one computes $\mathcal{L}^{*j}[\mathcal{M}]$, $0 \leq j \leq p$ and then one computes $\mathbb{M}, \mathbb{L}, \mathbb{R}$ via (63). The integer $p \geq 1$ is chosen such that $\frac{h^{p+1}}{(p+1)!}$ to be small enough.

Step 1. Taking $X = \mathcal{G}$ (introduced in (9)) one computes $\Pi_l(N-1)[\mathcal{G}], l = 1,2,3$ via (62). To this end one computes first $\mathcal{L}^{*j}[\mathcal{G}]$ using (64). The obtained values for $\Pi_l(N-1)[\mathcal{G}]$ are used in (54a) written for $k = N-1$ to compute $\tilde{X}(t_{N-1}^-)$. In the same time one computes $\tilde{X}(t_{N-1}) = \sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[\mathcal{G}]$.

Step k. $(k \geq 2)$**.** Assume that we have already computed the matrices $\tilde{X}(t_l^-)$ and $\tilde{X}(t_l)$ with $k+1 \leq l \leq N-1$. Using (64) with $X = \tilde{X}(t_{k+1}^-)$ we compute the iterations $\mathcal{L}^{*j}[\tilde{X}(t_{k+1}^-)]$, $j = 0,\ 1, ..., p$. Taking $X = \tilde{X}(t_{k+1}^-)$ one computes $\Pi_l(k)[\tilde{X}(t_{k+1}^-)], l = 1,2,3$ via (62). One computes $\tilde{X}(t_k^-)$ using (54a) written for the computed values of the involved matrices. One computes $\tilde{X}(t_k)$ by $\tilde{X}(t_k) = \sum_{j=0}^{p} \frac{h^j}{j!} \mathcal{L}^{*j}[\tilde{X}(t_{k+1}^-)]$.

The algorithm stops after the computation of $\tilde{X}(t_k^-)$ and $\tilde{X}(t_k)$ for $k = N-1, N-2, ..., 0$.

Remark 11 *For the computation of the matrices $\tilde{\tilde{X}}(t_k^-)$ and $\tilde{\tilde{X}}(t_k)$ one may use a modified version of the previous algorithm where the Step 0 is removed and the align (54b) is employed instead of (54a).*

Remark 12 *The value of the optimal performance provided by Theorem 7 in (36) can be computed by*

$$J(\tilde{u}, x_0) = x_0^T \tilde{X}_{11}(0^-)x_0 + \sum_{k=0}^{N-1}\sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!}\mathbf{Tr}[(\mathcal{L}_{11}^{*j}[\tilde{X}(t_{k+1}^-)] + \frac{h}{j+2}\mathcal{L}_{11}^{*j}[\mathcal{M}])B_v V B_v^T] \tag{65}$$

and the value of the optimal performance provided by Theorem 10 in (50) can be computed by

$$J(u_{\Psi\phi}, x_0) = x_0^T \tilde{\tilde{X}}_{11}(0^-)x_0 + \sum_{k=0}^{N-1}\sum_{j=0}^{p} \frac{h^{j+1}}{(j+1)!} \times \mathbf{Tr}[\mathcal{L}_{11}^{*j}[\tilde{\tilde{X}}(t_{k+1}^-)]B_v V B_v^T], \tag{66}$$

where $\tilde{X}_{11}(0^-)$, $\tilde{\tilde{X}}(0^-)$, $\mathcal{L}_{11}^{*j}[\tilde{X}(t_{k+1}^-)]$, $\mathcal{L}_{11}^{*j}[\tilde{\tilde{X}}(t_{k+1}^-)]$, $\mathcal{L}_{11}^{*j}[\mathcal{M}]$ are the block components formed by the first n rows and n columns of the matrices $\tilde{X}(0^-), \tilde{\tilde{X}}(0^-), \mathcal{L}^{*j}[\tilde{X}(t_{k+1}^-)], \mathcal{L}^{*j}[\tilde{\tilde{X}}(t_{k+1}^-)]$ and $\mathcal{L}^{*j}[\mathcal{M}]$.

5 Numerical Example

Let us consider the classic example of a fourth-order model representing a nominal model for the CE150 helicopter model described by [6]. The matrices occurring in (1) are shown in the following.

$$A_0(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -14.5076 & 68.5210 & -2.0568 & 0 \\ 0 & -25 & 0 & -10 \end{pmatrix},$$

$$B_0(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 25 \end{pmatrix}, \; B_v(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_1(t) = 0.1A_0(t), \; B_1(t) = 0.1B_0(t).$$

It should be noted that 10% of the magnitudes of the state and input matrices can be represented by the Wiener process based on stochastic perturbations as the state and control dependent noise. Furthermore, the matrices G, $M(t)$, $R(t)$ and the sampling and terminal times are given above.

$$G = \mathbf{diag}\begin{pmatrix} 1 & 2 & 3 & 3 \end{pmatrix},\ M(t) = I_4,\ R(t) = 2,$$
$$t_{k+1} - t_k = 0.01,\ t_N = \tau = 10.$$

Using the values of A_k, B_k, B_v, $k = 0,\ 1$, G, R, M we simulate the optimal control problem for controlled system (1) and cost performance (2). We compute the values $\tilde{X}(t_k)$ and $\tilde{X}(t_k^-)$ following steps described in section C. In addition we apply (34) to find $\tilde{F}(k)$ for $k = 0,\ 1, \ldots, N-1$. These values are described here:

$$\begin{aligned}
\tilde{F}(0) &= \begin{pmatrix} 0.8355 & -3.2093 & -0.4772 & -1.6973 \end{pmatrix},\\
\tilde{F}(1) &= \begin{pmatrix} 0.6611 & -2.3518 & -0.4077 & -1.6874 \end{pmatrix},\\
\tilde{F}(2) &= \begin{pmatrix} 0.5047 & -1.5811 & -0.3378 & -1.6787 \end{pmatrix},\\
\tilde{F}(3) &= \begin{pmatrix} 0.3684 & -0.9079 & -0.2695 & -1.6714 \end{pmatrix},\\
\tilde{F}(4) &= \begin{pmatrix} 0.2537 & -0.3389 & -0.2043 & -1.6657 \end{pmatrix},\\
\tilde{F}(5) &= \begin{pmatrix} 0.1611 & 0.1229 & -0.1442 & -1.6623 \end{pmatrix},\\
\tilde{F}(6) &= \begin{pmatrix} 0.0906 & 0.4775 & -0.0912 & -1.6620 \end{pmatrix},\\
\tilde{F}(7) &= \begin{pmatrix} 0.0416 & 0.7263 & -0.0479 & -1.6664 \end{pmatrix},\\
\tilde{F}(8) &= \begin{pmatrix} 0.0127 & 0.8733 & -0.0174 & -1.6776 \end{pmatrix},\\
\tilde{F}(9) &= \begin{pmatrix} 0.0011 & 0.9296 & -0.0023 & -1.6973 \end{pmatrix}.
\end{aligned}$$

In addition, we obtain the optimal performance of (36) via (65), i.e. $J(\tilde{u}, x_0) = 0.7983$.

6 CONCLUSION

A LQ optimal control problem for the sampled-data stochastic systems has been discussed. By using the jump system theory, it has been shown that the drawback that the measurement states are only accessed can be avoided. Since the controller can be designed by solving the Riccati differential and difference equations, it is worth pointing out that it is easy to construct the proposed controller for the practical stochastic systems.

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