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On the linear quadratic optimization problems and associated Riccati equations for systems modeled by Ito linear differential equations

Vasile Dragan
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
Research Unit 2, POBox 1-764, RO-014700
Bucharest, Romania
vasile.dragan@imar.ro

Abstract. In this paper a class of stochastic linear quadratic problems is investigated. Conditions which guarantee the existence of a solution of the associated Riccati differential equation and algebraic Riccati equation of stochastic control are formulated and some properties are derived in the cases of finite time horizon and infinite time horizon.

Key Words: Riccati differential equations, algebraic Riccati equations, stochastic linear quadratic control problems, stabilizing solution.

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1 Introduction

Several problems concerning the properties of solutions of matrix Riccati differential (difference) equations were investigated by researchers both in control theory and in the domain of differential (difference) equations. The existence of the global solutions of the Riccati equations has been related for a long time to the Jacobi sufficient conditions in the calculus of variations. In 1960, Kalman [1] obtained global existence results for a matrix Riccati differential equation under the so-called controllability and observability conditions in control theory; the result was related to the so-called linearquadratic optimization problem. Later on, Wonham [2, 3] extended the result to the framework of stochastic control and introduced the so-called Riccati equations of stochastic control.

There exist an enormous number of papers and monographs that deal with different problems concerning matrix Riccati differential (difference) equations, both in deterministic and stochastic frameworks. For the convenience of the readers, we refer here to the monographs [4, 5] and the references therein. These monographs contain a lot of aspects concerning both symmetric and non-symmetric Riccati differential (difference) equations as well algebraic Riccati equations. Different contributions to the problem of stability and stabilization for the systems modeled by stochastic differential equations can be found in [6, 7, 8, 9].

The problem of the existence of the stabilizing solution of an algebraic Riccati equation associated to a linear quadratic control problem described by a system of linear Itô differential equation with state and control dependent noise and a quadratic cost functional with indefinite sign was investigated in [14]. In [10, 11] the problem of the existence and uniqueness of the bounded and stabilizing solution of a Riccati differential equation arising in connection with a linear quadratic optimization problem described by a system of Itô differential equations perturbed by a Markov process was studied beside the existence of the maximal solution and the minimal solution. The existence of the stabilizing solution of the Riccati differential equations arising in connection with stochastic H_∞ control problems was studied in [15] for time varying case and in [16] for the time invariant case. Recently, the problem of the existence of the bounded and stabilizing solution of the Riccati differential equation involved in the description of the equilibrium strategy of a zero sum linear quadratic differential game was considered in [17] in the case when the dynamic system is modeled by a system of Itô differential equations with coefficients perturbed by a Markov process with a finite number of states.

The numerical computation of the stabilizing solution of a Riccati differential (algebraic) equation arising in the stochastic H_∞ control problem and stochastic zero sum linear quadratic differential games was investigated in [18, 19, 20, 21] where different iterative procedures were proposed to approximate the stabilizing solution of the considered Riccati equations.

The aim of the present work is to emphasize the role of some solutions with given terminal values or the role of the stabilizing solution of a matrix Riccati equation associated to a linear quadratic control problem described by a system of Itô differential equations with state and control dependent noise and a quadratic cost functional with indefinite sign. We shall display the sign conditions which need to be satisfied by the quadratic part of the Riccati differential equations or algebraic Riccati equations in order to be possible to solve a minimization problem, a maximization problem or a zero sum linear quadratic differential game. We shall see that in the case when the diffusion part of the controlled system contains a control dependent term, the sign of the weights matrices of the cost functional is less relevant to guarantee the well posedness of the linear quadratic control problems. We shall provide also conditions which guarantee the existence of the global solutions on a given interval $[0, T]$ as well as the existence of the stabilizing solution of a Riccati differential algebraic equation involved in the solution of a minimization, maximization problems or in construction of the equilibrium strategy of a zero sum linear quadratic stochastic differential game.

2 A class of stochastic linear quadratic problems. Model description

In this paper we consider the linear quadratic control problem (LQP) described by a linear controlled system of the form

$$dx(t) = (A_0x(t) + B_0u(t))dt + (A_1x(t) + B_1u(t))dw(t) \tag{1}$$

$t \geq 0$, $x(0) = x_0 \in \mathbb{R}^n$ and by the quadratic cost performance

$$J(T, x_0; u) = E[x_u^T(T) M_f x_u(T) + \int_0^T \begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} M & L \\ L^T & R \end{pmatrix} \begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix} dt] \tag{2}$$

in the case of a control problem on a compact interval (finite time horizon) and

$$J(\infty, x_0; u) = E[\int_0^\infty \begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} M & L \\ L^T & R \end{pmatrix} \begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix} dt] \tag{3}$$

in the case of infinite time horizon i.e. $t \in \mathbb{R}_+ = [0, \infty)$. Here, $x(t) \in \mathbb{R}^n$ are the state parameters of the controlled process and $u(t) \in \mathbb{R}^m$ are the input parameters which incorporates control parameters and/or some external perturbations which may affect the process which must be controlled.

In the sequel, for the sake of simplicity, $u(t)$ will be named control vectors. In (1) $\{w(t)\}_{t\geq 0}$ is one dimensional standard brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, that is

$$E[w(t)] = 0, \;\; E[(w(t) - w(s))^2] = t - s \quad \forall \quad t \geq s \geq 0.$$

Throughout E denotes the mathematical expectation and the superscript T stands for the transposition operation of a vector or a matrix.

In the case of (2) and (3) $x_u(\cdot)$ is the solution of the problem with given initial values (1) corresponding to the input $u(\cdot)$. Here A_j, B_j, $j = 0, 1$, M, M_f, L, R are given matrices with real entries of compatible dimensions. We assume that $M = M^T$, $M_f = M_f^T$, $R = R^T$.

If $\mathcal{I} \subset [0, \infty)$ is a compact interval, then $L_w^2(\mathcal{I}, \mathbb{R}^p)$ is the linear space of the measurable stochastic processes, $y : \mathcal{I} \times \Omega \to \mathbb{R}^p$ which are adapted to the filtration generated by the brownian motion $\{w(t)\}_{t\geq 0}$ and additionally $E[\int_{\mathcal{I}} |y(t)|^2 dt] < \infty$.

By $L_w^2(\mathbb{R}_+, \mathbb{R}^m)$ we denote the linear space of stochastic processes $u : [0, \infty) \times \Omega \to \mathbb{R}^m$ having the properties:

a) $u \in L_w^2(\mathcal{I}, \mathbb{R}^m)$ for any compact interval $\mathcal{I} \subset [0, \infty]$.

b) $E[\int_0^\infty |u(t)|^2 dt] < \infty$.

Throughout $|\cdot|$ denotes the euclidian norm of a vector that is $|u|^2 = u^T u$. One shows that for each $T > 0$ and for arbitrary $u \in L_w^2([0, T], \mathbb{R}^m)$ the Itô differential equation (1) has a unique solution $x_u = \{x_u(t)\}_{0\leq t\leq T}$ starting from x_0 at initial time $t = 0$ and additionally $x_u \in L_w^2([0, T], \mathbb{R}^n)$.

Let us introduce the class of admissible controls:

(i) in the case of a LQP described by the system (1) and the cost functional (2), the class of admissible controls $\mathcal{U}_{adm}(T, x_0)$ coincides with $L_w^2([0, T], \mathbb{R}^m)$;

(ii) in the case of a LQP on infinite time horizon, the class of admissible controls $\mathcal{U}_{adm}(\infty, x_0)$ consists of all stochastic processes $u \in L_w^2(\mathbb{R}_+, \mathbb{R}^m)$ with the property that $x_u \in L_w^2(\mathbb{R}_+, \mathbb{R}^m)$ and

$$\lim_{t\to\infty} E[x_u^T(t) x_u(t)] = 0. \tag{4}$$

If $\mathbb{J}(x_0, u)$ is one of the cost functionals $J(T, x_0; u)$ or $J(\infty, x_0; u)$ defined in (2) and (3) respectively, and $\mathbb{U}_{adm}(x_0)$ is one of the class of admissible controls $\mathcal{U}_{adm}(T, x_0)$, or $\mathcal{U}_{adm}(\infty, x_0)$ respectively, we can state three main kinds of linear quadratic optimization problems (LQOPs).

$LQOP_1$) (minimization) Given $x_0 \in \mathbb{R}^n$ find $\tilde{u} \in \mathbb{U}_{adm}(x_0)$ such that

$$\mathbb{J}(x_0, \tilde{u}) = \min_{u \in \mathbb{U}_{adm}(x_0)} \mathbb{J}(x_0, u).$$

$LQOP_2$) (maximization) Given $x_0 \in \mathbb{R}^n$ find a control $\tilde{u} \in \mathbb{U}_{adm}(x_0)$ such that

$$\mathbb{J}(x_0, \tilde{u}) = \max_{u \in \mathbb{U}_{adm}(x_0)} \mathbb{J}(x_0, u).$$

$LQOP_3$) (zero sum LQ differential game) One assumes that there exists a partition of the admissible inputs according with certain physical interpretation of the meaning of the inputs. So, we assume that $u(t) = (u_1(t), u_2(t))$, $u_k(t) \in \mathbb{R}^{m_k}$, where $m_k \geq 1$, $k = 1, 2$ and $m_1 + m_2 = m$, m_1 and m_2 do not depend upon $u \in \mathbb{U}_{adm}(x_0)$. In few words, the optimal control problem may be stated as follows:

Given $x_0 \in \mathbb{R}^n$, let find $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in \mathbb{U}_{adm}(x_0)$ with the property that

$$\mathbb{J}(x_0; u_1, \tilde{u}_2) \leq \mathbb{J}(x_0; \tilde{u}_1, \tilde{u}_2) \leq \mathbb{J}(x_0; \tilde{u}_1, u_2) \tag{5}$$

for all $u_1(t), u_2(t)$, such that $(u_1(t), \tilde{u}_2(t))$ and $(\tilde{u}_1(t), u_2(t))$ lie in $\mathbb{U}_{adm}(x_0)$.

Remark 1 *From (5) one sees that the goal of the input u_1 is to maximize the cost functional, while the aim of the input u_2 is to minimize the cost $\mathbb{J}(x_0; u_1, u_2)$. The input $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ satisfying (5) will be a saddle point of the cost $\mathbb{J}(x_0; \cdot, \cdot)$. Often $(\tilde{u}_1, \tilde{u}_2)$ satisfying (5) is named zero sum Nash equilibrium strategy.*

Remark 2 *For each $x_0 \in \mathbb{R}^n$ and a given pair $(\mathbb{J}(x_0; \cdot, \cdot), \mathbb{U}_{adm}(x_0))$ only one of the three optimal control problems stated above is possible.*

3 Riccati differential equations and algebraic Riccati equations of stochastic control

Let $\mathcal{S}_n \subset \mathbb{R}^{n \times n}$ be the linear space of symmetric matrices with real entries. On $\mathcal{S}_n$ we consider the matrix differential equation

$$\frac{d}{dt} X(t) + \mathcal{R}[X(t)] = 0 \tag{6}$$

where $X \to \mathcal{R}[X] : Dom(\mathcal{R}) \subset \mathcal{S}_n \to \mathcal{S}_n$ is described by

$$\begin{aligned} \mathcal{R}[X] &= A_0^T X + X A_0 + A_1^T X A_1 + M - \\ &- (X B_0 + A_1^T X B_1 + L)(R + B_1^T X B_1)^{-1}(B_0^T X + B_1^T X A_1 + L^T) \end{aligned} \tag{7}$$

where $Dom(\mathcal{R}) = \{X \in \mathcal{S}_n | R + B_1^T X B_1 \text{ is invertible}\}$.

If $A_1 = 0$, $B_1 = 0$, the nonlinear differential equation (6)-(7) reduces to the well known Riccati differential equation arising in connection with linear quadratic control problems in the deterministic framework. That is why, in the sequel (6)-(7) will be named Riccati differential equation (RDE) of stochastic control.

If $X : \mathcal{I} \subset \mathbb{R} \to \mathcal{S}_n$ is a solution of RDE (6)-(7) we define the gain matrix associated to this solution by

$$F^X(t) = -(R + B_1^T X(t) B_1)^{-1}(B_0^T X(t) + B_1^T X(t) A_1 + L^T), \quad t \in \mathcal{I}. \tag{8}$$

A. The case of finite time horizon

Let $X_T(t)$ be the solution of RDE (6)-(7) satisfying the given terminal condition $X_T(T) = M_f$, where M_f is the weight matrix arising in (2). Some conditions which guarantee the fact that the solution $X_T(t)$ is well defined for any $t \in [0, T]$ will be presented later.

Now, assuming that $X_T(t)$ is well defined on the whole interval $[0, T]$ and involving the Itô formula for computation of the stochastic differential of some random processes (see e.g. [12]) we may rewrite (2) as:

$$\begin{aligned} J(T, x_0; u) &= x_0^T X_T(0) x_0 + E\,[\int_0^T (u(t) - F_T(t) x_u(t))^T \\ &\times (R + B_1^T X_T(t) B_1)(u(t) - F_T(t) x_u(t)) dt] \end{aligned} \tag{9}$$

$\forall u \in \mathcal{U}_{adm}(T, x_0)$, $F_T(t)$ is computed as in (8) with $X(t)$ replaced by $X_T(t)$, $0 \le t \le T$.

Let $x_T(t)$ be the solution of the problem with given initial values

$$dx(t) = (A_0 + B_0 F_T(t)) x(t) dt + (A_1 + B_1 F_T(t)) x(t) dw(t), \;\; x_T(0) = x_0. \tag{10}$$

Setting

$$u_T(t) = F_T(t) x_T(t), \;\; 0 \le t \le T \tag{11}$$

we remark that $u_T \in \mathcal{U}_{adm}(T, x_0)$ and from the uniqueness of the solution of (10) we deduce that $x_{u_T}(t)$ coincides with $x_T(t)$. From (9) we obtain $J(T, x_0; u_T) = x_0^T X_T(0) x_0$.

Let us introduce the notation

$$\Re(X_T(t)) = R + B_1^T X_T(t) B_1. \tag{12}$$

We have:

Theorem 3 *Assume that the solution $X_T(\cdot)$ of the RDE (6)-(7) satisfying the terminal condition $X_T(T) = M_f$ is well defined for any $t \in [0, T]$. Under these conditions the following hold:*

a) If

$$\Re(X_T(t)) > 0 \tag{13}$$

for all $t \in [0, T]$ then for each $x_0 \in \mathbb{R}^n$, the optimal control problem described by the controlled system (1) and the performance (2) has a unique solution which is given by

$$\tilde{u}(t) = u_T(t) = F_T(t) x_T(t), \quad t \in [0, T], \tag{14}$$

$x_T(t)$ being the solution of the problem (10).

The minimal value of the cost functional (2) is given by

$$J(T, x_0; \tilde{u}) = x_0^T X_T(0) x_0. \tag{15}$$

b) If

$$\mathfrak{R}(X_T(t)) < 0 \tag{16}$$

for all $t \in [0, T]$, then for any $x_0 \in \mathbb{R}^n$ the control $\tilde{u}(t)$ defined in (14) is the unique admissible control which solves the optimal control problem of type LQOP2 described by the controlled system (1) and the cost functional (2). In this case (15) provides the maximal value of the performance criterion (2).

c) Assume that $\mathfrak{R}(X_T(t))$ has m_1 negative eigenvalues and m_2 positive eigenvalues where $m_k \geq 1, k = 1, 2$ not depend upon t and $m_1 + m_2 = m$.

Let $\begin{pmatrix} \mathfrak{R}_{11}(X_T(t)) & \mathfrak{R}_{12}(X_T(t)) \\ \mathfrak{R}_{12}^T(X_T(t)) & \mathfrak{R}_{22}(X_T(t)) \end{pmatrix}$ be the partition of the matrix $\mathfrak{R}(X_T(t))$.

We also consider the partition of the inputs $u(t)$ and $u_T(t)$ in the form $u(t) = (u_1(t), u_2(t))$, $u_T(t) = (u_{1T}(t), u_{2T}(t))$ where $u_k(t)$, $u_{kT}(t) \in \mathbb{R}^{m_k}$, $k = 1, 2$. If $\mathfrak{R}_{11}(X_T(t)) < 0$ and $\mathfrak{R}_{22}(X_T(t)) > 0$ for all $0 \leq t \leq T$, then $(u_{1T}(t), u_{2T}(t))$ is a saddle point (zero sum Nash equilibrium strategy) of LQ differential game described by the system (1) and the cost functional (2), i.e.

$$J(T, x_0; u_1, u_{2T}) \leq J(T, x_0; u_{1T}, u_{2T}) \leq J(T, x_0; u_{1T}, u_2)$$

for all $u_k \in L_w^2([0, T], \mathbb{R}^{m_k})$, $k = 1, 2$.

Remark 4 *The result stated in Theorem 3 points out the fact that in each of the three optimal control problems considered here, the gain matrix of the optimal control is constructed based on the solution of the RDE (6)-(7) satisfying the terminal condition $X_T(T) = M_f$. Even if $X_T(t)$ is unique it can contribute to the computation of the optimal control of the three distinct optimization problems. This fact is possible due to the properties of the sign of the Kernel matrix (12). That is why, in practice, in order to solve an LQ optimization problem it is looking for the solution $X_T(\cdot)$ of RDE (6)-(7) satisfying the terminal condition $X_T(T) = M_f$ and the Kernel matrix (12) has an imposed sign.*

B. The case of the infinite time horizon

Let $X : [0, \infty) \to \mathcal{S}_n$ be a bounded solution of RDE (6)-(7). Based on the Itô formula we obtain the following form of (3)

$$J(\infty, x_0; u) = x_0^T X(0) x_0 + E[\int_0^\infty (u(t) - F^X(t) x_u(t))^T (R + B_1^T X(t) B_1)(u(t) - F^X(t) x_u(t)) dt] \tag{17}$$

for all $u \in \mathcal{U}_{adm}(\infty, x_0)$, $x_0 \in \mathbb{R}^n$, where $F^X(t)$ is the gain matrix associated to the solution X via (8).

Let $x^X(t)$, $t \in [0, \infty)$ be the solution of the problem with given initial values:

$$dx(t) = (A_0 + B_0 F^X(t))x(t)dt + (A_1 + B_1 F^X(t))x(t)dw(t), \quad x^X(0) = x_0. \tag{18}$$

We set

$$u^X(t) = F^X(t)x^X(t). \tag{19}$$

From the uniqueness of the solution of the problem with given initial values (18) we may infer that $x_{u^X}(t)$ coincides with $x^X(t)$, $t \geq 0$.

Note: Unlike the controls of type (11) from the finite time horizon case, in the case of controls of type (19) we cannot conclude that they are admissible controls because, we cannot be sure that $\lim_{t\to\infty} E[|x_{u^X}(t)|^2] = 0$.

This fact entitles us to define a special kind of bounded solution of RDE (6)-(7).

Definition 5 *We say that a solution $X : [0, \infty) \to \mathcal{S}_n$ is a stabilizing solution of RDE (6)-(7) if all trajectories $x^X(t, x_0)$ of the corresponding closed-loop system of type (18) satisfy $\lim_{t\to\infty} E[|x^X(t, x_0)|^2] = 0$ for all $x_0 \in \mathbb{R}^n$*

The next result is a special case of that proved in Theorem 5.6.5 from [13].

Theorem 6 *The RDE (6)-(7) has at most one bounded and stabilizing solution. Moreover, the unique bounded and stabilizing solution of RDE (6)-(7) if it exists, is constant, and therefore it solves the following algebraic Riccati equation of stochastic control (SARE):*

$$\begin{aligned} A_0^T X + XA_0 + A_1^T XA_1 + M - \\ -(XB_0 + A_1 XB_1 + L)(R + B_1^T XB_1)^{-1}(B_0^T X + B_1^T XA_1 + L^T) = 0. \end{aligned} \tag{20}$$

In the next, we shall denote $\tilde{X}$ the unique stabilizing solution of SARE (20) if it exists. We also set

$$\tilde{F} = F^{\tilde{X}} = -(R + B_1^T \tilde{X} B_1)^{-1}(B_0^T \tilde{X} + B_1^T \tilde{X} A_1 + L^T). \tag{21}$$

The analogous of the control (19) is

$$\tilde{u}(t) = \tilde{F}\tilde{x}(t) \tag{22}$$

where $\tilde{x}(t) = \tilde{x}(t, x_0)$, $t \geq 0$, is the solution of the problem with given initial values

$$\begin{aligned} dx(t) &= (A_0 + B_0\tilde{F})x(t)dt + (A_1 + B_1\tilde{F})x(t)dw(t) \\ \tilde{x}(0) &= x_0. \end{aligned} \tag{23}$$

From the definition of the stabilizing solution of (20) we know that the trajectories of (23) satisfy $\lim_{t\to\infty} E[|\tilde{x}(t, x_0)|^2] = 0$, $\forall x_0 \in \mathbb{R}^n$.

Hence, the controls (22) lie in $\mathcal{U}_{adm}(\infty, x_0)$ for all $x_0 \in \mathbb{R}^n$.

When $X(t)$ is replaced by the stabilizing solution $\tilde{X}$, (17) becomes

$$J(\infty, x_0; u) = x_0^T \tilde{X} x_0 + E[\int_0^\infty (u(t) - \tilde{F} x_u(t))^T (R + B_1^T \tilde{X} B_1)(u(t) - \tilde{F} x_u(t))dt] \qquad (24)$$

for all $u \in \mathcal{U}_{adm}(\infty; x_0)$, $x_0 \in \mathbb{R}^n$.

The identity (24) allows us to obtain:

Theorem 7 *If $\tilde{X}$ is the unique stabilizing solution of SARE (20), then the following possibilities occur:*

a) if $R + B_1^T \tilde{X} B_1 > 0$ then for any $x_0 \in \mathbb{R}^n$ the linear quadratic optimal control of type $LQOP_1$ described by the cost functional (3) and the class of admissible controls $\mathcal{U}_{adm}(\infty; x_0)$ has a unique solution which is given by the control (22). The minimal value of the cost functional (3) is

$$J(\infty, x_0; \tilde{u}) = x_0^T \tilde{X} x_0. \qquad (25)$$

b) If $R + B_1^T \tilde{X} B_1 < 0$ then for any $x_0 \in \mathbb{R}^n$ the optimal control problem of type $LQOP_2$ described by the cost functional (3) and the class of admissible controls $\mathcal{U}_{adm}(\infty, x_0)$ has a unique solution which is given by the control (22). In this case (25) provides the maximal value of the cost functional (3).

c) Assume that the matrix $R + B_1^T \tilde{X} B_1$ has m_1 negative eigenvalues and m_2 positive eigenvalues, $m_k \geq 1$, $k = 1, 2$ and $m_1 + m_2 = m$. Assume also that we have the partition

$$R + B_1^T \tilde{X} B_1 = \begin{pmatrix} R_{11} + B_{11}^T \tilde{X} B_{11} & R_{12} + B_{11} \tilde{X} B_{12} \\ R_{12}^T + B_{12}^T \tilde{X} B_{11} & R_{22} + B_{12}^T \tilde{X} B_{12} \end{pmatrix}$$

such that $R_{11} + B_{11}^T \tilde{X} B_{11} < 0$ and $R_{22} + B_{12}^T \tilde{X} B_{12} > 0$. Under these conditions the control (22) admits the partition $\tilde{u}(t) = (\tilde{u}_1(t), \tilde{u}_2(t))$, $\tilde{u}_k(t) \in \mathbb{R}^{m_k}$ and it is a saddle point of the cost functional (3) i.e. $J(\infty, x_0; u_1, \tilde{u}_2) \leq x_0^T \tilde{X} x_0 = J(\infty, x_0; \tilde{u}_1, \tilde{u}_2) \leq J(\infty, x_0; \tilde{u}_1, u_2)$ for all $u_k \in L_w^2(\mathbb{R}_+, \mathbb{R}^{m_k})$ such that $(u_1, \tilde{u}_2)$ and $(\tilde{u}_1, u_2)$ lie in $\mathcal{U}_{adm}(\infty; x_0)$.

Remark 8 *The result stated in Theorem 7 emphasizes the role of the sign of the matrix $R + B_1^T \tilde{X} B_1$ to establish what kind of LQ optimal control problem can be solved involving the stabilizing solution of a given SARE (22). It is worth mentioning that in the case $B_1 \neq 0$ the sign of the weight matrix R is less relevant in the identification of the type of optimal LQ control problem which can be solved based on the stabilizing solution of the considered SARE.*

4 Conditions which guarantee the existence of a solution of RDE and ARE of stochastic control with desired properties

4.1 Riccati differential equations of stochastic control with definite sign of quadratic part

A) The case of finite time horizon

In this subsection we provide conditions which are equivalent to the well definiteness of the solution $X_T(\cdot)$ of RDE (6)-(7) on the whole interval $[0,T]$ with the property that $\Re(X_T(t))$ has definite sign for all t.

For a differential matrix valued function $Z : \mathcal{I} \subset \mathbb{R} \to \mathcal{S}_n$ we denote

$$\mathbb{D}(Z(t)) = \begin{pmatrix} \dot{Z}(t) + \mathcal{L}[Z(t)] + M & Z(t)B_0 + A_1^T Z(t)B_1 + L \\ B_0^T Z(t) + B_1^T Z(t)A_1 + L^T & \Re(Z(t)) \end{pmatrix} \tag{26}$$

where $Y \to \mathcal{L}[Y] : \mathcal{S}_n \to \mathcal{S}_n$,

$$\mathcal{L}[Y] = A_0^T Y + Y A_0 + A_1^T Y A_1 \tag{27}$$

and $\Re(Z(t))$ is computed as in (12) with $X_T(t)$ replaced by $Z(t)$, i.e. $\Re(Z(t)) = R + B_1^T Z(t) B_1$.

The linear operator $\mathcal{L}$ introduced in (27) is the Lyapunov type operator associated to the system of linear Itô differential equations (obtained from (1) for $B_j = 0, j = 0, 1$)

$$dx(t) = A_0 x(t)dt + A_1 x(t)dw(t). \tag{28}$$

Definition 9 *We say that the system (28) is asymptotically stable in mean square (ASMS) or equivalently the pair (A_0, A_1) is stable if its solutions are satisfying $\lim_{t\to\infty} E[|x(t,x_0)|^2] = 0$ for all $x_0 \in \mathbb{R}^n$.*

The main properties of the operator $\mathcal{L}$ involved in this work are summarized below:

Proposition 10 *(i) the system (28) is ASMS if and only if the eigenvalues of the operator $\mathcal{L}$ are placed in the half plane $\mathbb{C}_- = \{z \in \mathbb{C} | Rez < 0\}$;*

(ii) the operator $\mathcal{L}$ defines a positive evolution on the Hilbert space $\mathcal{S}_n$, i.e., the solutions of the differential equation

$$\dot{X}(t) = \mathcal{L}[X(t)] \tag{29}$$

are positive semidefinite matrices for all $t \geq 0$, if $X(0) \geq 0$;

(iii) if $Y(t; T, H)$ is the solution of the affine differential equation on $\mathcal{S}_n$

$$\dot{Y}(t) + \mathcal{L}[Y(t)] + \Lambda(t) = 0 \tag{30}$$

satisfying the condition $Y(T) = H$ ($\Lambda(\cdot)$ being a continuous matrix valued function) then $Y(t; T, H) \geq 0 \quad \forall t \leq T$ if $H \geq 0$ and $\Lambda(t) \geq 0 \quad \forall t \leq T$.

Proof: (i) Using the representation formula given in Proposition 3.1.3 from [13] one shows that (A_0, A_1) is stable if and only if the solutions of the differential equation (29) tend to 0 for $t \to \infty$. This happens if and only if the eigenvalues of the operator $\mathcal{L}$ are in $\mathbb{C}_-$.

(ii) We have $\mathcal{L} = \mathcal{L}^0 + \Pi$ where $\mathcal{L}^0[X] = A_0^T X + X A_0$ and $\Pi[X] = A_1^T X A_1$; $e^{\mathcal{L}^0 t}[X] = e^{A_0^T t} X e^{A_0 t} \geq 0$, $\forall t \geq 0$ if $X \geq 0$. $\Pi[X] \geq 0$ if $X \geq 0$. The conclusion follows from Corollary 2.2.6 from [13].

(iii) The solutions of (30) have the representation

$$Y(t;T,H) = e^{\mathcal{L}(T-t)}[H] + \int_t^T e^{\mathcal{L}(s-t)}[\Lambda(s)]\, ds$$

and the conclusion follows from (ii). □

Using the equivalence from (i) in Proposition 10 in the case of the closed loop system (23) we obtain:

Corollary 11 *The following are equivalent:*

(i) $\tilde{X}$ is the stabilizing solution of SARE (20);

(ii) the eigenvalues of the linear operator $\mathcal{L}_{\tilde{F}} : \mathcal{S}_n \to \mathcal{S}_n$ are placed in the half plane $\mathbb{C}_-$, where

$$X \to \mathcal{L}_{\tilde{F}}[X] = (A_0 + B_0\tilde{F})^T X + X(A_0 + B_0\tilde{F}) + (A_1 + B_1\tilde{F})^T X(A_1 + B_1\tilde{F})$$

$\tilde{F}$ being defined in (21);

(iii) the eigenvalues of the linear operator $\mathcal{R}'[\tilde{X}]$ are located in the half place $\mathbb{C}_-$, $\mathcal{R}'[\tilde{X}]$ being the Freche derivative of the Riccati operator $\mathcal{R}[\cdot]$ defined in (7) computed in $X = \tilde{X}$.

The next result provides conditions which are equivalent to the well definiteness of the solution $X_T(\cdot)$ of RDE (6)-(7) satisfying conditions (13) and (16) from Theorem 3.

Theorem 12 *a) Let $\hat{\mathcal{I}}(T) \subset [0,T]$ be the maximal interval where the solution $X_T(\cdot)$ of RDE (6)-(7) taking the terminal value $X_T(T) = M_f$ is defined and satisfies the sign condition $R + B_1^T X_T(t) B_1 > 0 \quad \forall t \in \hat{\mathcal{I}}(T)$. Then the following are equivalent:*

(i) $\hat{\mathcal{I}}(T) = [0,T]$

(ii) there exists a differentiable function $\hat{Z} : [0,T] \to \mathcal{S}_n$ satisfying the conditions:

$$\begin{aligned} \mathbb{D}[\hat{Z}(t)] &\geq 0, \\ R + B_1^T Z(t) B_1 &> 0 \end{aligned} \tag{31}$$

$t \in [0,T]$ and $M_f \geq \hat{Z}(T)$.

b) Let $\check{\mathcal{I}}(T) \subset [0,T]$ be the maximal interval where the solution $X_T(\cdot)$ of RDE (6)-(7) taking the terminal value $X_T(T) = M_f$ is well defined and satisfies the sign condition $R + B_1^T X_T(t) B_1 < 0$ $\forall t \in \check{\mathcal{I}}(T)$.

Then the following are equivalent:

(i) $\check{\mathcal{I}}(T) = [0,T]$;

(ii) there exists a differentiable function $\check{Z} : [0,T] \to \mathcal{S}_n$ satisfying the conditions:

$$\begin{aligned} \mathbb{D}[\check{Z}(t)] &\leq 0 \\ R + B_1^T \check{Z}(t) B_1 &< 0 \end{aligned} \tag{32}$$

$t \in [0,T]$ and $M_f \leq \check{Z}(T)$.

Proof: One shows that $\hat{Z}(t) \leq X_T(t) \leq Y_T(t)$ for all $t \in \hat{\mathcal{I}}(T)$ if $\hat{Z}(\cdot)$ satisfies (31) and $Y_T(t) \leq X_T(t) \leq \check{Z}(t)$ for all $t \in \check{\mathcal{I}}(T)$ if $\check{Z}(\cdot)$ satisfies (32). In both cases $Y_T(\cdot)$ is the solution with given terminal values $\dot{Y}(t) + A_0^T Y(t) + Y(t)A_0 + A_1^T Y(t)A_1 + M = 0$, $Y(T) = M_f$. Thus one obtains that $X_T(t)$ is bounded on its maximal interval of definition. Hence, it can be extended to the whole interval $[0, T]$. □

Remark 13 *a) The conditions (31) and (32), respectively, compensate the absence of the additional assumptions regarding the sign of the weights M and R. If $\hat{Z}(t) = 0$, $0 \leq t \leq T$, satisfies (31) we obtain the case of LQOP with defined sign:*

$$\begin{pmatrix} M & L \\ L^T & R \end{pmatrix} \geq 0, quad R > 0. \tag{33}$$

In this case $X_T(t) \geq 0 = \hat{Z}(t)$ and so the condition $R + B_1^T X(t) B_1 > 0$ is automatically satisfied.

b) Since the coefficients A_j, B_j, $j = 0, 1$, M, L, R are constant functions it can be checked the solvability of (31) and (32) respectively, looking for constant functions $\hat{Z}$, $\check{Z}$ respectively.

B. The case of infinite time horizon

Definition 14 *We say that the system (1) is stochastic stabilizable if there exists a feedback gain matrix $F \in \mathbb{R}^{m\times n}$ such that the control $u(t) = Fx(t)$ stabilizes that system, i.e. the closed-loop system $dx(t) = (A_0 + B_0F)x(t)dt + (A_1 + B_1F)x(t)dw(t)$ is ASMS.*

Proposition 15 *The following are equivalent:*
(i) the system (1) is stochastic stabilizable;
(ii) there exist the matrices $Z \in \mathcal{S}_n$, $W \in \mathbb{R}^{m\times n}$ solving the following LMI:

$$\begin{pmatrix} \Upsilon(Z, W) & A_1 Z + B_1 W & \\ Z A_1^T + W^T B_1^T & -Z & < 0 \end{pmatrix} \tag{34}$$

where $\Upsilon(Z, W) = A_0 Z + Z A_0^T + B_0 W + W^T B_0^T$. If (Z, W) is a solution of LMI (34) then the control $u(t) = W Z^{-1} x(t)$ stabilizes the system (1).

For proof we refer to [14].

Theorem 16 *The following are equivalent:*
(i) SARE (20) has a stabilizing solution $\tilde{X}$ satisfying the sign condition $R + B_1^T \tilde{X} B_1 > 0$;
(ii) the system (1) is stochastic stabilizable and there exists $\hat{Z}$ satisfying the LMI $\mathbb{D}[\hat{Z}] >$ 0.

If the stabilizing solution $\tilde{X}$ exists it satisfies the maximality condition

$$\tilde{X} = \max\{X \in \mathcal{S}_n | \mathbb{D}[X] \geq 0, \;\; R + B_1^T X B_1 > 0\}. \tag{35}$$

Remark 17 *In the statement of Theorem 16 no assumption regarding the sign of the weights matrices is done and hence, no information regarding the sign of the stabilizing solution $\tilde{X}$ is available.*

In the special case described by (33) the stabilizing solution is positive semidefinite matrix. Moreover, in this case, we have $\tilde{X} \geq \tilde{\tilde{X}} \geq 0$ where $\tilde{\tilde{X}}$ is the minimal positive semidefinite solution of (20). In general $\tilde{X} \neq \tilde{\tilde{X}}$.

This can be viewed from the following example:
Example. Consider $n = 2$, $m = 1$. Choose

$$\begin{aligned} A_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, A_1 = I_2,\ B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ M &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = 1, L = 0. \end{aligned}$$

The maximal solution of the Riccati equation is

$$\tilde{X} = \begin{bmatrix} 8 & -21 \\ -21 & 63 \end{bmatrix} > 0$$

and the minimal solution is

$$\tilde{\tilde{X}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

Obviously in this case $\tilde{X} \neq \tilde{\tilde{X}}$.

Note. In the absence of some additional assumptions, the positive semidefinite solution of (20) is not equivalent with the stabilizing solution of that equation.

The condition (33) is equivalent to $R > 0$ and $M - LR^{-1}L^T \geq 0$.
Let C be such that $C^TC = M - LR^{-1}L^T$.
We set $\hat{A}_j = A_j - B_jR^{-1}L^T$, $j = 0, 1$ and consider the system

$$\begin{aligned} dx(t) &= \hat{A}_0x(t)dt + \hat{A}_1x(t)dw(t) \\ y(t) &= Cx(t). \end{aligned} \tag{36}$$

Definition 18 *We say that the system (36) is stochastic detectable if there exists a matrix of compatible dimensions H, such that the system*

$$dx(t) = (\hat{A}_0 + HC)x(t)dt + \hat{A}_1x(t)dw(t) \tag{37}$$

is ASMS.

Proposition 19 *The following are equivalent:*
(i) the system (36) is stochastic detectable;
(ii) there exists $Y = Y^T > 0$ and V solving the following LMI:

$$\hat{A}_0^TY + Y\hat{A}_0 + C^TV^T + VC + \hat{A}_1^TY\hat{A}_1 < 0.$$

If (Y, V) is a solution of this LMI, then $H = Y^{-1}V$ satisfies the condition from Definition 4.

Proposition 20 *The following are true:*

(i) if the auxiliary system (36) is stochastic detectable then any positive semidefinite solution of SARE (20) is a stabilizing one;

(ii) if the system (1) is stochastic stabilizable and the auxiliary system (36) is stochastic detectable, then the SARE (20) has a unique positive semidefinite solution and that solution is stabilizing.

Note. To obtain the uniqueness of positive semidefinite solution of SARE (20) we have added the assumption of stochastic detectability.

Regarding the existence of the stabilizing solution of SARE (20) satisfying the sign condition $R + B_1^T \tilde{X} B_1 < 0$ we have:

Theorem 21 *The following are equivalent:*

(i) the SARE (20) has a stabilizing solution $\tilde{X}$ satisfying the sign condition $R+B_1^T \tilde{X} B_1 < 0$;

(ii) the system (1) is stochastic stabilizable and there exist $\check{Z} \in \mathcal{S}_n$ satisfying $\mathbb{D}[\check{Z}] < 0$.

If the stabilizing solution $\tilde{X}$ of (20) satisfying $R + B_1^T \tilde{X} B_1 < 0$ exists, then it satisfies the following minimality condition

$$\tilde{X} = \min\{X \in \mathcal{S}_n | \mathbb{D}[X] \leq 0, \quad R + B_1^T X B_1 < 0\}.$$

Proof: $\tilde{X}$ is stabilizing solution of SARE (20) satisfying the sign condition $R + B_1^T \tilde{X} B_1 < 0$ if and only if $\tilde{Y} = -\tilde{X}$ is the stabilizing solution of the SARE

$$A_0^T Y + Y A_0 + A_1^T Y A_1 - (Y B_0 + A_1^T Y B_1 - L)(-R + B_1^T Y B_1)^{-1}(B_0^T Y + B_1^T Y A_1 - L^T) = 0$$

satisfying condition $-R + B_1^T Y B_1 > 0$. The conclusion is obtained applying Theorem 16 in the case of SARE (37). □

4.2 Riccati equations with indefinite sign of their quadratic parts

In this subsection we consider the case when the solution $X(\cdot)$ of RDE (6)-(7) are such that the matrices $R + B_1^T X(t) B_1$ have m_1 negative eigenvalues and m_2 positive eigenvalues, $m_k \geq 1$, $k = 1, 2$, do not depend upon t and $m_1 + m_2 = m$.

Setting $u(t) = (u_1(t), u_2(t))$, $u_k(t) \in \mathbb{R}^{m_k}$, $k = 1, 2$, we rewrite (1)-(3) in the form:

$$dx(t) = (A_0 x(t) + B_{01} u_1(t) + B_{02} u_2(t))dt + (A_1 x(t) + B_{11} u_1(t) + B_{12} u_2(t))dw(t) \tag{38}$$

$x(0) = x_0, B_{jk} \in \mathbb{R}^{n \times m_k}, k = 1, 2, j = 0, 1,$

$$\begin{aligned} J(T, x_0; u_1, u_2) = E\,[x_u^T(T) M_f x_u(T) + \int_0^T \begin{pmatrix} x_u^T(t) & u_1^T(t) & u_2^T(t) \end{pmatrix} \\ \times \mathbb{Q} \begin{pmatrix} x_u^T(t) & u_1^T(t) & u_2^T(t) \end{pmatrix}^T dt] \end{aligned} \tag{39}$$

$$J(\infty, x_0; u_1, u_2) = E[\int_0^\infty \left(x_u^T(t)\ \ u_1^T(t)\ \ u_2^T(t)\right) \mathbb{Q} \left(x_u^T(t)\ \ u_1^T(t)\ \ u_2^T(t)\right)^T dt] \tag{40}$$

where $\mathbb{Q} = \begin{pmatrix} M & L_1 & L_2 \\ L_1^T & R_{11} & R_{12} \\ L_2^T & R_{12}^T & R_{22} \end{pmatrix}$, with $L_k \in \mathbb{R}^{n \times m_k}$, $k = 1, 2$, $R_{lk} \in \mathbb{R}^{m_l \times m_k}$, $l, k = 1, 2$.

Let $K : [0, T] \to \mathbb{R}^{m_2 \times n}$ be a continuous matrix valued function. Plugging $u_2(t) = K(t)x(t)$ in (38) and (39) we obtain

$$\begin{aligned} dx(t) &= (A_{0K}(t)x(t) + B_{01}u_1(t))dt + (A_{1K}(t)x(t) + B_{11}u_1(t))dw(t) \\ x(0) &= x_0 \end{aligned} \tag{41}$$

and

$$J_K(T, x_0; u_1) = E[x_{u_1}^T(T) M_f x_{u_1}(T) + \int_0^T \begin{pmatrix} x_{u_1}(t) \\ u_1(t) \end{pmatrix}^T \begin{pmatrix} M_K(t) & L_K(t) \\ L_K^T(t) & R_{11} \end{pmatrix} \begin{pmatrix} x_{u_1}(t) \\ u_1(t) \end{pmatrix} dt] \tag{42}$$

where $A_{jk}(t) = A_j + B_{j2}K(t)$, $j = 0, 1$, $M_K(t) = \begin{pmatrix} I_n \\ K(t) \end{pmatrix}^T \begin{pmatrix} M & L_2 \\ L_2^T & R_{22} \end{pmatrix} \begin{pmatrix} I_n \\ K(t) \end{pmatrix}$, $L_K(t) = L_1 + K^T(t)R_{12}^T$, $0 \le t \le T$, $x_{u_1}(\cdot)$ is the solution of the problem with given initial values (41).

The corresponding RDE associated to the controlled system (41) and the cost functional (42) is:

$$\begin{aligned} &\dot{Y}(t) + A_{0K}^T(t)Y(t) + Y(t)A_{0K}(t) + A_{1K}^T(t)Y(t)A_{1K}(t) + M_K(t) - \\ &-(Y(t)B_{01} + A_{1K}^T(t)Y(t)B_{11} + R_K(t))(R_{11} + B_{11}^T Y(t)B_{11})^{-1} \times \\ &\times (B_{01}^T Y(t) + B_{11}^T Y(t)A_{1K}(t) + L_K^T(t)) = 0. \end{aligned} \tag{43}$$

In order to facilitate the statement of the next result we shall introduce several notations. So, $\mathfrak{K}_T$ denotes the set of continuous matrix valued functions $K : [0, T] \to \mathbb{R}^{m_2 \times n}$ with the property that the solution $Y_{KT}(\cdot)$ of the corresponding RDE (43) satisfying the terminal condition $Y_{KT}(T) = M_f$ is well defined on the whole interval $[0, T]$ and satisfies the sign condition $R_{11} + B_{11}^T Y_{KT}(t)B_{11} < 0$, $\forall t \in [0, T]$.

Set

$$\mathbb{D}_2[Z(t)] = \begin{pmatrix} \dot{Z}(t) + \mathcal{L}[Z(t)] + M & Z(t)B_{02} + A_1^T Z(t)B_{12} + L_2 \\ B_{02}^T Z(t) + B_{12}^T Z(t)A_1 + L_2^T & L_{22} + B_{12}^T Z(t)B_{12} \end{pmatrix} \tag{44}$$

where $\mathcal{L}[\cdot]$ is the Lyapunov type operator associated to the pair (A_0, A_1) by (27) and $Z : [0, T] \to \mathcal{S}_n$ is an arbitrary differentiable matrix valued function.

Remark 22 *The matrix valued function $\mathbb{D}_2[Z(t)]$ is, in fact, the matrix valued function of type (26) but associated to the triple $((A_0, A_1), (B_{02}, B_{12}), (M, L_2, R_{22}))$. To this triple we associate the RDE:*

$$\dot{X}(t) + \mathcal{R}_2[X(t)] = 0 \tag{45}$$

where

$$\begin{aligned}&\mathcal{R}_2[X] = A_0^T X + X A_0 + A_1^T X A_1 + M\\&-(XB_{02} + A_1^T X B_{12} + L_2)(R_{22} + B_{12}^T X B_{12})^{-1}(B_{02}^T X + B_{12}^T X A_1 + L_2^T).\end{aligned} \tag{46}$$

The following auxiliary results will be involved in the developments in this section.

Lemma 23 *Assume that* $X \in \mathcal{S}_n$ *is such that the matrix* $R + B_1^T X B_1 \in \mathcal{S}_m$ *has* m_1 *negative eigenvalues and* m_2 *positive eigenvalues,* $m_k \geq 1$, $k = 1, 2$, $m_1 + m_2 = m$.

a) If $\begin{pmatrix} I_{m_1} & 0 \end{pmatrix}(R + B_1^T X B_1)\begin{pmatrix} I_{m_1} & 0 \end{pmatrix}^T < 0$ *then we have the following factorization* $R + B_1^T X B_1 = \tilde{V}^T(X)\Im\tilde{V}(X)$ *where* $\tilde{V}(X) = \begin{pmatrix} \tilde{V}_{11}(X) & \tilde{V}_{12}(X) \\ 0 & \tilde{V}_{22}(X) \end{pmatrix}$ *with* $\tilde{V}_{jj}(X) = \tilde{V}_{jj}^T(X) > 0$, $j = 1, 2$, *and* $\Im = \begin{pmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2} \end{pmatrix}$.

b) If $\begin{pmatrix} 0 & I_{m_2} \end{pmatrix}(R + B_1^T X B_1)\begin{pmatrix} 0 & I_{m_2} \end{pmatrix}^T > 0$ *then we have* $R + B_1^T X B_1 = V^T(X)\Im V(X)$ *where* $V(X) = \begin{pmatrix} V_{11}(X) & 0 \\ V_{21}(X) & V_{22}(X) \end{pmatrix}$ *with* $V_{jj}(X) = V_{jj}^T(X) > 0$, $j = 1, 2$, *and* $\Im$ *is the same as before.*

Proof: is done by direct calculation. The details are omitted. □

Lemma 24 *If* $W \in \mathbb{R}^{m \times n}$ *is an arbitrary matrix we have*

$$\mathcal{R}[X] = (A_0 + B_0 W)^T X + X(A_0 + B_0 W) + (A_1 + B_1 W)^T X (A_1 + B_1 W) -$$
$$-(W - F^X)^T (R + B_1^T X B_1)(W - F^X) + \mathbb{M}_W$$

for all $X \in Dom\mathcal{R}$, *where* F^X *is defined as in (8) with* $X(t)$ *replaced by* X *and* $\mathbb{M}_W = \begin{pmatrix} I_n \\ W \end{pmatrix}^T \begin{pmatrix} M & L \\ L^T & R \end{pmatrix} \begin{pmatrix} I_n \\ W \end{pmatrix}$.

Proof: It follows by direct calculations. □

It is worth mentioning that $M_K(t)$ involved in (42) and (43) is the special case of $\mathbb{M}_W$ for $W = \begin{pmatrix} 0 \\ K(t) \end{pmatrix}$.

For each $0 \leq t_0 < T < \infty$ we introduce the performance criterion

$$J(T, t_0, x_0; u_1, u_2) = E[x_u^T(T) M_f x_u(T) + \int_{t_0}^{T} \begin{pmatrix} x_u^T(t) & u_1^T(t) & u_2^T(t) \end{pmatrix} \mathbb{Q} \begin{pmatrix} x_u^T(t) & u_1^T(t) & u_2^T(t) \end{pmatrix}^T dt]$$

$x_u(t)$ being the solution of the differential equation (38) corresponding to the input $u = (u_1, u_2) \in L_w^2([t_0, T], \mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$ and satisfying the initial condition $x_u(t_0) = x_0 \in \mathbb{R}^n$. □

Lemma 25 *Let* $X(t)$ *be the solution of RDE (6)-(7) satisfying* $X_T(T) = M_f$. *If* $t_0 < T$ *is in the interval of definition of the solution* $(X_T(\cdot))$, *we have*

$$J(T, t_0, x_0; u_1, u_2) = x_0^T X_T(t_0) x_0 + J(T, t_0, 0; u_1 - u_{1T}, u_2 - u_{2T})$$

for all $(u_1, u_2) \in L_w^2([t_0, T], \mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$ *where* $u_{1T}(t) = \begin{pmatrix} I_{m_1} & 0 \end{pmatrix} F_T(t) x_T(t)$, $u_{2T}(t) = \begin{pmatrix} 0 & I_{m_2} \end{pmatrix} F_T(t) x_T(t)$, $F_T(t)$ *being computed as in (8) for* $x(t)$ *replaced by* $x_T(t)$ *and* $x_T(t)$ *is the solution of the corresponding closed-loop system of type (10) and satisfies* $x_T(t_0) = x_0$.

Proof: It follows the line of the proof of Lemma 4.3 from [15]. □

Now, we are in position to state a result which provides a set of conditions equivalent to the global existence of solution of RDE in the case of a zero sum LQ differential game.

Theorem 26 *Assume that*

$$\mathbb{Q}_2 \triangleq \begin{pmatrix} M & L_2 \\ L_2^T & R_{22} \end{pmatrix} \geq 0. \tag{47}$$

For a $T > 0$ we denote $\mathcal{I}(T)$ the maximal interval $\mathcal{I}(T) \subset [0,T]$ where the solution $X_T(t)$ of RDE (6)-(7) satisfying $X_T(T) = M_f$ is well defined and satisfies the sign conditions

$$\begin{aligned} R_{11} + B_{11}^T X_T(t) B_{11} < 0 \\ R_{22} + B_{12}^T X_T(t) B_{12} > 0. \end{aligned} \tag{48}$$

Then the following are equivalent:

(i) $\mathcal{I}(T) = [0,T]$;

(ii) the set $\mathfrak{K}$ is not empty and there exists a differentiable function $Z : [0,T] \to \mathcal{S}_n$ satisfying

$$\begin{aligned} \mathbb{D}_2[Z(t)] \geq 0 \\ R_{22} + B_{12}^T Z(t) B_{12} > 0. \end{aligned} \tag{49}$$

Proof: $(i) \Rightarrow (ii)$ If $X_T(t)$ is well defined for any $t \in [0,T]$, means that $F_T(t)$ is well defined on the interval $[0,T]$. We set $F_{1T}(t) = \begin{pmatrix} I_{m_1} & 0 \end{pmatrix} F_T(t)$ and $F_{2T}(t) = \begin{pmatrix} 0 & I_{m_2} \end{pmatrix} F_T(t)$. Employing Lemma 24 with $W = \begin{pmatrix} 0 \\ F_{2T}(t) \end{pmatrix}$ combined with the factorization provided by Lemma 23 (a) in the case of matrix $R + B_1^T X_T B_1$ we may infer that $F_{2T}(\cdot) \in \mathfrak{K}_T$. On the other hand, the factorization provided by Lemma 23 (b) applied in the case of $R + B_1^T X_T(t) B_1$ allows us to deduce that $X_T(t)$ satisfies (49).

In order to prove $(ii) \Rightarrow (i)$ let us remark that if $K(\cdot) \in \mathfrak{K}_T$ and if $Y_T(\cdot)$ is the corresponding solution of RDE (43) satisfying $Y_T(T) = M_f$, then, one obtains

$$X_T(t) \leq Y_T(t), \qquad \forall \quad t \in \mathcal{I}(T).$$

To this end one may use Lemma 25 together with (47). Also, one shows that $X_T(t) \geq Z(t)$, $t \in \mathcal{I}(T)$. Thus, one obtains that $X_T(t)$ is bounded and it can be extended to $[0,T]$. □

B. The case of infinite time horizon

Let $\mathfrak{K}_\infty$ be the set of gain matrices $K \in \mathbb{R}^{m_2 \times n}$ with the properties:

a) the system of linear *Itô* differential equations

$$dx(t) = (A_0 + B_{02}K)x(t)dt + (A_1 + B_{12}K)x(t)dw(t)$$

is ASMS;

b) the associated ARE

$$\begin{aligned} A_{0K}^T Y + Y A_{0K} + A_{1K}^T Y A_{1K} + M_K - (Y B_{01} + A_{1K}^T Y B_{11} + L_K)(L_{11} + \\ + B_{11}^T Y B_{11})^{-1}(B_{01}^T Y + B_{11}^T Y A_{1K} + L_K^T) = 0 \end{aligned} \tag{50}$$

has a stabilizing solution $\tilde{Y}_K$ satisfying the sign condition

$$R_{11} + B_{11}^T \tilde{Y}_K B_{11} < 0 \tag{51}$$

where $A_{jk} = A_j + B_{j2}K$, $j = 0, 1$, $M_K = \begin{pmatrix} I_n \\ K \end{pmatrix}^T \begin{pmatrix} M & L_2 \\ L_2^T & R_{22} \end{pmatrix} \begin{pmatrix} I_n \\ K \end{pmatrix}$, $L_K = L_1 + K^T R_{12}^T$.

Let us assume that the conditions hold

$$R_{22} > 0, \quad M - L_2 R_{22}^{-1} L_2^T \geq 0. \tag{52}$$

Consider the auxiliary system

$$\begin{aligned} dx(t) &= (A_0 - B_{02}R_{22}^{-1}L_2^T)x(t)dt + (A_1 - B_{12}R_{22}^{-1}L_2^T)x(t)dw(t) \\ y(t) &= Cx(t) \end{aligned} \tag{53}$$

where C is obtained from the factorization $M - L_2 R_{22}^{-1} L_2^T = C^T C$.

The main result of this section is:

Theorem 27 *Assume:*

a) the conditions (52) are fulfilled;

b) the auxiliary system (53) is stochastic detectable;

c) the set $\mathfrak{K}_\infty$ is not empty.

Under these conditions the SARE (20) has a stabilizing solution $\tilde{X} \geq 0$ having the following additional properties:

(i) $R_{11} + B_{11}^T \tilde{X} B_{11} < 0, \quad R_{22} + B_{12}^T \tilde{X} B_{12} > 0$.

(ii) $\tilde{X} \leq \hat{X}$ for any solution $\hat{X} \geq 0$ of (20) with the property $R + B_1^T \hat{X} B_1$ has m_1 negative eigenvalues and m_2 positive eigenvalues.

Proof: For each $T > 0$ let $X_T(\cdot)$ be the solution of RDE (6)-(7) satisfying the terminal condition $X_T(T) = 0$. Let $K \in \mathfrak{K}_\infty$. Since $\mathfrak{K}_\infty \subset \mathfrak{K}_T$ for all $T > 0$ we may conclude according with the result proved in Theorem **??** that $X_T(t)$ is well defined for any $t \in [0, T]$ and all $T > 0$. Also, one shows that

$$0 \leq X_{T_1}(t) \leq X_{T_2}(t) \leq \tilde{Y}_K \tag{54}$$

for all $0 \leq t \leq T_1 < T_2$, $\tilde{Y}_K$ being the stabilizing solution of the corresponding SARE of type (50).

We set $\tilde{X} = \lim_{T \to \infty} X_T(0)$. From (54) we deduce that $\tilde{X}$ is well defined.

From (51) and (54) one obtains that

$$R_{11} + B_{11}^T X_T(t) B_{11} \leq R_{11} + B_{11}^T \tilde{Y}_K B_{11} < 0.$$

So, we obtain that $R_{11} + B_{11}^T \tilde{X} B_{11} < 0$. The condition $R_{22} + B_{12}^T \tilde{X} B_{12} > 0$ is automatically satisfied because $R_{22} > 0$ and $\tilde{X} \geq 0$. We set

$$\tilde{F} = -(R + B_1^T \tilde{X} B_1)^{-1} (B_0^T \tilde{X} + B_1^T \tilde{X} A_1 + L^T)$$

and $\tilde{F}_1 = \begin{pmatrix} I_{m_1} & 0 \end{pmatrix} \tilde{F}$, $\tilde{F}_2 = \begin{pmatrix} 0 & I_{m_2} \end{pmatrix} \tilde{F}$.

Employing the assumption b) one shows that the system

$$dx(t) = (A_0 + B_{02}\tilde{F}_2)x(t)dt + (A_1 + B_{12}\tilde{F}_2)x(t)dw(t) \tag{55}$$

is ASMS. In order to obtain that $\tilde{X}$ is just the stabilizing solution of (20) we need to show that the system

$$dx(t) = (A_0 + B_{01}\tilde{F}_1 + B_{02}\tilde{F}_2)x(t)dt + (A_1 + B_{11}\tilde{F}_1 + B_{12}\tilde{F}_2)x(t)dw(t) \tag{56}$$

is ASMS. To this end we first show that there exists $c > 0$ such that

$$E[\int_0^\infty |\tilde{F}_1\tilde{x}(t,x_0)|^2 dt] \leq c|x_0|^2$$

$\forall x_0 \in \mathbb{R}^n$, $\tilde{x}(t,x_0)$ being the solution of (56) satisfying $\tilde{x}(0,x_0) = x_0$.

In this way, we may rewrite (56) in the form

$$d\tilde{x}(t) = [(A_0 + B_0\tilde{F}_2)\tilde{x}(t) + f_0(t)]dt + [(A_1 + B_{12}\tilde{F}_2)\tilde{x}(t) + f_1(t)]dw(t) \tag{57}$$

where $f_k(t) = B_{k1}\tilde{F}_1\tilde{x}(t) \in L^2_w(\mathbb{R}_+, \mathbb{R}^{m_k})$, $k = 1,2$, $\tilde{x}(t) = \tilde{x}(t,x_0)$ is an arbitrary solution of (56).

Invoking the property of ASMS of (**??**) we may obtain that the trajectories of the system (57) satisfy

$$\lim_{t\to\infty} E[|\tilde{x}(t,x_0)|^2] = 0 \qquad \forall \ x_0 \in \mathbb{R}^n.$$

This means that the system (56) is ASMS. This ends the proof. □

Remark 28 *One may shows that if SARE (20) has a solution $\tilde{X} \geq 0$ satisfying the sign conditions $R_{11} + B_{11}^T\tilde{X}B_{11} < 0$ and $R_{22} + B_{12}^T\tilde{X}B_{12} > 0$ then $\tilde{K} = \tilde{F}_2$ lies in $\mathfrak{K}_\infty$. Hence, the assumption c) in the statement of Theorem 16 is a necessary condition too, for the existence of the stabilizing solution of (20) satisfying the sign conditions from above.*

Remark 29 *In order to check if the auxiliary system (53) is stochastic detectable, we have to test the feasibility of the following LMI*

$$(A_0 - B_{02}R_{22}^{-1}L_2^T)^T Z + C^T W^T + Z(A_0 - B_{02}R_{22}^{-1}L_2^T) + WC + (A_1 - B_{12}R_{22}^{-1}L_2^T)^T Z(A_1 - B_{12}R_{22}^{-1}L_2^T) < 0$$

$$Z > 0$$

The next result may be used to test if the set $\mathfrak{K}_\infty$ is not empty.

Proposition 30 *Under the assumption that $\begin{pmatrix} M & L_2 \\ L_2^T & R_{22} \end{pmatrix} \geq 0$ the following are equivalent:*

(i) the set of feedback gains $\mathfrak{K}_\infty$ is not empty;

(ii) there exist matrices $Z \in \mathcal{S}_n$, $\Gamma \in \mathbb{R}^{m_2 \times n}$, satisfying the following LMI

$$\begin{pmatrix} \Xi_0(Z,\Gamma) & \Xi_1(Z,\Gamma) & \Xi_2(Z,\Gamma) & \Xi_3(Z,\Gamma) \\ \Xi_1^T(Z,\Gamma) & R_{11} & B_{11}^T & 0 \\ \Xi_2^T(Z,\Gamma) & B_{11} & -Z & 0 \\ \Xi_3^T(Z,\Gamma) & 0 & 0 & -I \end{pmatrix} < 0 \tag{58}$$

where we have denoted

$$\Xi_0(Z,\Gamma) = A_0 Z + B_{02}\Gamma + Z A_0^T + \Gamma^T B_{02}^T$$

$$\Xi_1(Z,\Gamma) = B_{01} + Z L_1 + \Gamma^T R_{12}^T$$

$$\Xi_2(Z,\Gamma) = Z A_1^T + \Gamma^T B_{12}^T$$

$$\Xi_3(Z,\Gamma) = \begin{pmatrix} Z & \Gamma^T \end{pmatrix} \Theta^T$$

Θ^T *being obtained from the factorization* $\Theta^T\Theta = \begin{pmatrix} M & L_2 \\ L_2^T & R_{22} \end{pmatrix}$.

If (Z,Γ) is a solution of LMI (58) then $K = \Gamma Z^{-1} \in \mathfrak{K}_\infty$.

Proof: Based on Theorem 21, we deduce that $K \in \mathfrak{K}_\infty$ if and only if there exists $Y > 0$ satisfying the LMI

$$\begin{pmatrix} \Lambda_{11}(Y) & \Lambda_{12}(Y) \\ \Lambda_{12}^T(Y) & R_{11} + B_{11}^T Y B_{11} \end{pmatrix} < 0 \tag{59}$$

where

$$\Lambda_{11}(Y) = (A_0 + B_{02}K)^T Y + Y(A_0 + B_{02}K) + (A_1 + B_{12}K)^T Y (A_1 + B_{12}K) + M_K$$

$$\Lambda_{12}(Y) = Y B_{01} + (A_1 + B_{12}K)^T Y B_{11}.$$

Using the Schur complement technique one shows that (59) is equivalent to (58) with $Z = Y^{-1}$ and $\Gamma = KZ$. □

Remark 31 *If $\mathbb{Q}_2$ is not positive semidefinite we may perform the decomposition $\mathbb{Q}_2 = \mathbb{Q}_2^+ - \mathbb{Q}_2^-$ with $\mathbb{Q}_2^+ \geq 0$ and $\mathbb{Q}_2^- \geq 0$. One may check that the LMI of the form (58) with Θ obtained from the factorization $\Theta^T\Theta = \mathbb{Q}_2^+$ is a sufficient condition which guarantees that $\mathfrak{K}_\infty$ is not empty.*

References

[1] R. Kalman, Contribution on the theory of optimal control, *Bult. Soc. Math. Mex. Mexicana*, 5(1), 1960, 102–119.

[2] W.M. Wonham, On a matrix Riccati equation of stochastic control, *SIAM J. Control*, 6, 1968, 681–697.

[3] W.M. Wonham, Random differential equations in control theory, in *Probabilistic Methods in Applied Mathematics*, A.T. Bharucha-Reid, ed., vol. 2, Academic Press, New York/London, 1970.

[4] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Ricati Equations in Control and Systems Theory*, Birkhauser, Basel, 2003

[5] P.Lancaster, L.Rodman, *Algebraic Riccati Equations*, Calderon Press, Oxford, 1995.

[6] Y. Kats, A.A. Martynyuk, *Stability and Stabilization of Nonlinear Systems with Random Structure*, Taylor and Francis, London, 2002.

[7] X. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Process Appl.*, 79, 1999, 45–67.

[8] P.V. Pakshin, V.A. Ugrinovskii, Stochastic problems of absolute stability, *Automat Rem.Contr.*, 67, 2006, 1811–1846.

[9] C. Yuan, J. Lygeros, Stabilization of a class of stochastic differential equations with Markovian switching, *Syst. Control Lett.*, 54, 2005, 819–833.

[10] V. Dragan, T. Morozan, Systems of matrix rational differential equations arising in connection with linear stochastic systems with Markovian jumping, *Journal of differential equations*, 194, 1, 2003, 1–38.

[11] V. Dragan, G. Freiling, A. Hochhaus, T. Morozan, A class of nonlinear differential equations on the space of symmetric matrices, *Electron. J. Diff. Equ.*, 2004, 96, 2004, 1–48.

[12] B. Oxendal, *Stochastic Differential Equations*, Springer, Berlin, 1998.

[13] V. Dragan, T. Morozan, A.M.Stoica, *Mathematical methods in robust control of linear stochastic systems*, Springer New- York, 2013, second edition.

[14] M. A. Rami, X. Y. Zhou, Linear matrix inequalities, Riccati equations, indefinite stochastic linear quadratic controls, *IEEE Trans. Autom. Control*, 45, 6, 2000, 1131–1143.

[15] V. Dragan, T. Morozan, Global solutions to a game-theoretic Riccati equation of stochastic control, *Journal of Differ. Equat.*, 138, 2, 1997, 328–350.

[16] I.R. Petersen, V.A. Ugrinovskii and A.V. Savkin, *Robust Control Design using H_∞ Methods*, Springer, Berlin, 2000.

[17] V.Dragan, T. Morozan, On the bounded and stabilizing solution of the generalized Riccati differential equations with periodic coefficients arising in connection with a zero sum linear quadratic stochastic differential game. *New Trends in Differential Equations, Control Theory and Optimization, editori: V. Barbu, C.G. Lefter, I.I. Vrabie, editura: World Scientific*, 2016, 81–93, ISBN: 978 9813142855, 2016.

[18] V. Dragan, I. Ivanov, Computation of the stabilizing solution of game theoretic Riccati equation arising in stochastic H_∞ control problems, *Numerical Algorithms*, 57, 3, 2011, 357–375.

[19] V. Dragan, I. Ivanov, A numerical procedure to compute the stabilizing solution of game theoretic Riccati equations of stochastic control, *Int. Journal of Control*, 84, 4, 2011, 783–800.

[20] Y. Feng, B. D. O. Anderson, An iterative algorithm to solve state perturbed stochastic algebraic Riccati equations in LQ zero-sum games, *Systems and Control Letters*, 59, 2010, 50–56.

[21] D. Vrabie, F. Lewis, Adaptive dynamic programming for online solution of a zero-sum differential game, *Journal Contr. Theory and Appl.,* 9, 3, 2011, 353–360.