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On a Perturbation Estimate for the Extreme Solution of the Matrix Equation $X - A^*\widehat{X}^{-1}A = Q$ *

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Abstract. In this paper a perturbation estimate for the unique positive definite solution X_+ of the matrix equation $X - A^*\widehat{X}^{-1}A = Q$ is discussed. In [1] a perturbation estimate has been obtained under the condition $\|\widehat{PX_+^{-1}}AP\| < 1$, where P is a positive definite matrix. Moreover, in [1] there is an open question "How to choose the matrix P, such that $\|\widehat{PX_+^{-1}}AP^{-1}\| < 1$". Here we give an answer to above question. The theoretical results are illustrated by several numerical examples. Different perturbation estimates are compared.

Key Words: nonlinear matrix equation, perturbation estimates.

1 Introduction

In this paper we consider a perturbation estimate for the matrix equation

$$X - A^*\widehat{X}^{-1}A = Q\,, \tag{1}$$

where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix},$$

$A_1, A_2, \ldots, A_m, Q$ are $n \times n$ complex matrices, Q is a Hermitian positive definite, A^* is the conjugate transpose of a matrix A and $\widehat{X}$ is $m \times m$ block diagonal matrix with X on its diagonal.

Eq. (1) can be written as

$$X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q\,, \tag{2}$$

also.

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Ran and Reurings [11] have investigated equation

$$X - A^*(\widehat{X} - C)^{-1}A = Q\,, \tag{3}$$

where $A \in \mathcal{C}^{mn\times n}$, $C \in \mathcal{H}^{mn}$ and $C \geq 0$. This equation has a unique positive definite solution X_+, satisfying $\widehat{X_+} > C$, under the condition $\widehat{Q} > C$ [11]. Equation (3) is connected with certain interpolation problems [11, 12, Chapter 7]. The perturbation analysis of equation (1) and equation (3) is executed in Yin and Fang [13], and Sun [14], respectively. In addition, there are many contributions in the literature to the solvability, numerical solutions, and perturbation analysis for the matrix equations $X - \sum_{i=1}^m A_i^* X^{\delta_i} A_i = Q$ [15, 16, 17, 18], $X - \sum_{i=1}^m A_i^* \mathcal{F}(X) A_i = Q$ [19], $A_0 + \sum_{i=1}^k \sigma_i A_i^* X^{p_i} A_i = 0$ [20, 21].

The unique positive definite solution X_+ of equation (1) (or equation (3)) we will call extreme.

Eq. (1) for $m = 1$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [2, 3]. It has been investigated for the existence a positive definite solution in [2, 4, 5], and it has been executed the perturbation analysis in [6, 7, 8, 9, 10].

Now we consider the perturbed equation

$$\tilde{X} - \tilde{A}^* \widehat{\tilde{X}}^{-1} \tilde{A} = \tilde{Q}\,, \tag{4}$$

where $\tilde{A} \in \mathcal{C}^{mn\times n}$ and $\tilde{Q}$ ($\tilde{Q} > 0$) are small perturbations of A and Q in (1), respectively.

Let $\tilde{X}_+$ be the extreme solution of Eq. (4). Denote $\Delta X_+ = \tilde{X}_+ - X_+$, $\Delta Q = \tilde{Q} - Q$ and $\Delta A = \tilde{A} - A$.

Yin and Fang [13] have derived the following result.

Theorem 1 ([13, Theorem 2.1]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (1) and (4), respectively. Denote*

$$b = 1 - \|A\|^2 \|X_+^{-1}\|^2 + \|X_+^{-1}\| \|\Delta Q\|\,,$$

$$c = \|\Delta Q\| + 2\,\|A\| \|X_+^{-1}\| \|\Delta A\| + \|X_+^{-1}\| \|\Delta A\|^2\,,$$

$$\textit{and} \quad D = b^2 - 4c\|X_+^{-1}\|.$$

If

$$\|A\|^2 \|X_+^{-1}\|^2 < 1 \quad \textit{and} \quad 2\,\|\Delta A\| + \|\Delta Q\| \leq \frac{(1 - \|A\| \|X_+^{-1}\|)^2}{\|X_+^{-1}\|}\,, \tag{5}$$

then the extreme solutions X_+ and $\tilde{X}_+$ the respective equations (1) and (4) satisfies

$$\|\Delta X_+\| \leq \frac{b - \sqrt{D}}{2\|X_+^{-1}\|} =: S_{err}\,. \tag{6}$$

In [1] Theorem 1 has been modified as follows.

Theorem 2 ([1, Theorem 2]) *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (1) and (4), respectively, P is a positive definite matrix. Denote $\alpha_p = \|\widehat{PX_+^{-1}}AP^{-1}\|$, $\beta_p = \|PX_+^{-1}P\|$, where X_+ is the extreme solution of equation (1),*

$$\begin{aligned} b_p = & \quad 1 - \alpha_p^2 + \beta_p \|P^{-1}\Delta Q P^{-1}\| \,, \\ c_p = & \quad \|P^{-1}\Delta Q P^{-1}\| + 2\alpha_p \|\widehat{P}^{-1}\Delta A P^{-1}\| + \beta_p \|\widehat{P}^{-1}\Delta A P^{-1}\|^2 \,. \end{aligned}$$

If

$$\alpha_p < 1 \quad and \quad 2\|\widehat{P}^{-1}\Delta A P^{-1}\| + \|P^{-1}\Delta Q P^{-1}\| \leq \frac{(1-\alpha_p)^2}{\beta_p}\,, \tag{7}$$

then $D_p = b_p^2 - 4c_p\beta_p \geq 0$ and

$$\|\Delta X_+\| \;\leq\; \|P\|^2 \frac{b_p - \sqrt{D_p}}{2\beta_p} \;=:\; S_{err}^P\,. \tag{8}$$

In [1, Example 1] Eq. (1) has been considered with $m = 2$ and matrices A and Q as follows:

$$A = \begin{pmatrix} -0.4326 & -1.1465 \\ -1.6665 & 1.1909 \\ 0.1253 & 1.1892 \\ 0.2877 & -0.0376 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.1376 & 0.0656 \\ 0.0656 & 0.5616 \end{pmatrix}$$

and an approximation of the extreme solution

$$X_+ \approx \begin{pmatrix} 1.1572575 & 0.01971555 \\ 0.01971555 & 3.3569583 \end{pmatrix}.$$

For this example we know that $\|X_+^{-1}\|\|A\| \approx 1.9443 > 1$, $\|\widehat{X_+^{-1}}A\| = 1.4926 > 1$, but $\alpha_p = \|\widehat{PX_+^{-1}}AP^{-1}\| \approx 0.9621$, where $P = \sqrt{Q} + 2\sqrt[4]{Q}$ [1]. Moreover, in [1] there is an open question "How to choose the matrix P, such that $\|\widehat{PX_+^{-1}}AP^{-1}\| < 1$". In the next section we give an answer to the previous question.

The aim of this paper is to show how to choose a matrix P such that Theorem 2 remains valid. We will prove a new theorem where we propose a special choice of the matrix P. In addition, we are executing some numerical experiments where we compare different perturbation estimates for the extreme solution to (1).

In this paper we exploit the following notations. $\mathcal{C}^{p\times q}$ denotes the set of $p \times q$ complex matrices and $\mathcal{H}^n$ denotes the set of $n \times n$ Hermitian matrices. For the matrices $A = (a_{ij})$ and B, the notation $A \otimes B = (a_{ij}B)$ stands for a Kronecker product. The notation $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If $A - B > 0$ (or $A - B \geq 0$) we write $A > B$ (or $A \geq B$). We denote the identity matrix of order n with I (or I_n). The symbols $\|\cdot\|$, $\|\cdot\|_F$ and $\rho(\cdot)$ denote the spectral norm, the Frobenius norm, and the spectral radius, respectively. For any matrix Z, we denote with $\widehat{Z}$ the $m \times m$ block diagonal matrix with Z on its diagonal, i.e. $\widehat{Z} = I_m \otimes Z$.

2 Perturbation estimates

Firstly, we note that

$$\rho\Big(\big(X_+^{-T} \otimes X_+^{-1}\big)\sum_{i=1}^{m} A_i^T \otimes A_i^*\Big) = \rho\Big(\sum_{i=1}^{m}(X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) < 1,$$

$$\rho\Big(\sum_{i=1}^{m}(X_+^{-1}A_i)^T \otimes (X_+^{-1}A_i)^*\Big) \leq \Big\|\sum_{i=1}^{m} A_i^* X_+^{-2} A_i\Big\| = \|\widehat{X_+^{-1}A}\|^2 .$$

Therefore, a problem in the application of Theorem 2 to appear when $\|\widehat{X_+^{-1}A}\| \geq 1$. But, if $\|AX_+^{-1}\| < 1$, then Theorem 2 is applicable with $P = X_+ =: P_2$. In this case the estimate S_{err}^{P} in (8) we denote $S_{err}^{P_2}$.

Now, we will prove that

$$\|\widehat{\sqrt{X_+^{-1}}A\sqrt{X_+^{-1}}}\| < 1 \tag{9}$$

for arbitrary coefficient matrix A and the extreme solution X_+ of equation (1).

From equation (1) we have

$$\begin{aligned} A^*\widehat{X_+^{-1}}A &= X_+ - Q \geq 0, \\ \sqrt{X_+^{-1}}A^*\widehat{X_+^{-1}}A\sqrt{X_+^{-1}} &= I - \sqrt{X_+^{-1}}Q\sqrt{X_+^{-1}} \geq 0. \end{aligned}$$

Since $\sqrt{X_+^{-1}}Q\sqrt{X_+^{-1}} > 0$, then

$$\sqrt{X_+^{-1}}A^*\widehat{X_+^{-1}}A\sqrt{X_+^{-1}} < I$$

which implies (9).

Therefore, Theorem 2 is applicable with $P = \sqrt{X_+} =: P_1$. Then the estimate S_{err}^{P} in (8) we denote $S_{err}^{P_1}$.

For alternative of Theorem 2 with $P = \sqrt{X_+}$ we obtain the following result.

Theorem 3 *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (1) and (4), respectively. Denote $\alpha_1 = \|\widehat{\sqrt{X_+^{-1}}A\sqrt{X_+^{-1}}}\|$, where X_+ is the extreme solution of equation (1),*

$$\begin{aligned} b_1 = &\quad 1 - \alpha_1^2 + \|X_+^{-1}\|\|\Delta Q\| , \\ c_1 = &\quad \|\Delta Q\| + 2\alpha_1\|\Delta A\| + \|X_+^{-1}\|\|\Delta A\|^2 . \end{aligned}$$

If

$$2\|\Delta A\| + \|\Delta Q\| \leq \frac{(1-\alpha_1)^2}{\|X_+^{-1}\|} , \tag{10}$$

then $D_1 = b_1^2 - 4c_1\|X_+^{-1}\| \geq 0$ and

$$\|\Delta X_+\| \ \leq \ \|X_+\|\frac{b_1 - \sqrt{D_1}}{2} \ =: \ S_{err}^{P_1'} . \tag{11}$$

Proof: The proof is like to the proof of Theorem 2 ([1, Theorem 2]). In [1] it is obtained

$$\begin{aligned}\Delta X_+ = \Delta Q \,+\, \Delta A^*\big(I_{n^2}+\widehat{X_+^{-1}}\widehat{\Delta X_+}\big)^{-1}\widehat{X_+^{-1}}(A+\Delta A)\\ -\, A^*\widehat{X_+^{-1}}\big(I_{n^2}+\widehat{\Delta X_+}\widehat{X_+^{-1}}\big)^{-1}\big(\widehat{\Delta X_+}\widehat{X_+^{-1}}A-\Delta A\big)\,. \end{aligned} \tag{12}$$

From (12), we have

$$\begin{aligned}\sqrt{X_+^{-1}}\Delta X_+\sqrt{X_+^{-1}} =& \sqrt{X_+^{-1}}\Delta Q\sqrt{X_+^{-1}}+\sqrt{X_+^{-1}}\Delta A^*\widehat{\sqrt{X_+^{-1}}}(I_{n^2}+\widehat{\sqrt{X_+^{-1}}}\widehat{\Delta X_+}\widehat{\sqrt{X_+^{-1}}})^{-1}\\ &\times \widehat{\sqrt{X_+^{-1}}}(A+\Delta A)\sqrt{X_+^{-1}}-\sqrt{X_+^{-1}}A^*\widehat{\sqrt{X_+^{-1}}}\\ &\times (I_{n^2}+\widehat{\sqrt{X_+^{-1}}}\widehat{\Delta X_+}\widehat{\sqrt{X_+^{-1}}})^{-1}\widehat{\sqrt{X_+^{-1}}}(\widehat{\Delta X_+}\widehat{X_+^{-1}}A-\Delta A)\sqrt{X_+^{-1}}.\end{aligned}$$

Consider a map $\mu\,:\,\mathcal{H}^n\to\mathcal{H}^n$ defined by the following way

$$\begin{aligned}\mu(M) =& \sqrt{X_+^{-1}}\Delta Q\sqrt{X_+^{-1}}+\sqrt{X_+^{-1}}\Delta A^*\widehat{\sqrt{X_+^{-1}}}\big(I_{n^2}+\widehat{M}\big)^{-1}\\ &\times\Big(\widehat{\sqrt{X_+^{-1}}}A\sqrt{X_+^{-1}}+\widehat{\sqrt{X_+^{-1}}}\Delta A\sqrt{X_+^{-1}}\Big)\\ &-\sqrt{X_+^{-1}}A^*\,\widehat{\sqrt{X_+^{-1}}}\big(I_{n^2}+\widehat{M}\big)^{-1}\Big(\widehat{M}\,\widehat{\sqrt{X_+^{-1}}}A\sqrt{X_+^{-1}}-\widehat{\sqrt{X_+^{-1}}}\Delta A\sqrt{X_+^{-1}}\Big).\end{aligned}$$

From the conditions of the theorem, we have

$$\begin{aligned}2\|\Delta A\|\|X_+^{-1}\|+\|\Delta Q\|\|X_+^{-1}\| &\le (1-2\alpha_1+\alpha_1^2)\,,\\ b_1 = 1-\alpha_1^2+\|X_+^{-1}\|\|\Delta Q\| &\le 2-2(\alpha_1+\|X_+^{-1}\|\|\Delta A\|)<2\,,\\ D_1 = b_1^2-4c_1\|X_+^{-1}\| &= b_1^2-4b_1+4-4\big(\alpha_1+\|X_+^{-1}\|\|\Delta A\|\big)^2\ge 0\,.\end{aligned} \tag{13}$$

Since $D_1\ge 0$, the quadratic equation

$$S^2-b_1S+c_1\|X_+^{-1}\|=0 \tag{14}$$

has two positive real roots. If $D_1>0$, then the smaller root is

$$S_1=\frac{b_1-\sqrt{D_1}}{2}\,.$$

If $D_1=0$, $S_1=\frac{b_1}{2}$ is the double root of (14).

We define

$$\mathcal{L}_{S_1}=\{M\in\mathcal{H}^n:\|M\|\le S_1\}.$$

For each $M\in\mathcal{L}_{S_1}$ by (13), we have

$$\|M\|\le S_1\le\frac{b_2}{2}<1\,.$$

Thus, the matrices $I + M$ and $I_{n^2} + \widehat{M}$ are nonsingular, $\mu(M)$ is a continuous map and

$$\|\big(I_{n^2} + \widehat{M}\big)^{-1}\| \leq \frac{1}{1 - \|M\|} \leq \frac{1}{1 - S_1}\,.$$

Denote $\Delta\alpha = \|\widehat{\sqrt{X_+^{-1}}\Delta A \sqrt{X_+^{-1}}}\|$. Then for each $M \in \mathcal{L}_{S_1}$, we obtain

$$\begin{aligned}\|\mu(M)\| &\leq \|\sqrt{X_+^{-1}}\Delta Q \sqrt{X_+^{-1}}\| + \Delta\alpha \frac{\alpha_1 + \Delta\alpha}{1 - S_1} + \alpha_1 \frac{\alpha_1 S_1 + \Delta\alpha}{1 - S_1} \\ &\leq \|X_+^{-1}\|\|\Delta Q\| + \frac{2\alpha_1 \|X_+^{-1}\|\|\Delta A\| + \|X_+^{-1}\|^2\|\Delta A\|^2 + S_1\alpha_1^2}{1 - S_1} \\ &= \frac{(1 - b_1)S_1 \;+\; c_1\|X_+^{-1}\|}{1 - S_1} = S_1\,.\end{aligned}$$

Thus $\mu(M) \in \mathcal{L}_{S_1}$ for every $M \in \mathcal{L}_{S_1}$, which means that $\mu(\mathcal{L}_{S_1}) \subset \mathcal{L}_{S_1}$. By the Schauder's fixed point theorem, there exists an $M_* \in \mathcal{L}_{S_1}$ such that $\mu(M_*) = M_*$. Hence there exists a Hermitian solution ΔX_* of the equation (12) for which

$$\|\sqrt{X_+^{-1}}\Delta X_* \sqrt{X_+^{-1}}\| \leq S_1$$

and

$$\|\Delta X_*\| \leq \|X_+\|\, S_1\,.$$

Let

$$\tilde{X}_* = X_+ + \Delta X_*\,. \tag{15}$$

Since X_+ is the positive definite solution of (1) and ΔX_* is a Hermitian solution of (12), then $\tilde{X}_*$ is a Hermitian solution of the perturbed equation (4). The positive definiteness of $\tilde{X}_*$ can be proved on same way as in Theorem 2. Thus, $\tilde{X}_* \equiv \tilde{X}_+$ and $\Delta X_* \equiv \Delta X_+$. □

Above we noted that if $\|AX_+^{-1}\| < 1$, then Theorem 2 is applicable with $P = X_+$ ($P_2 := X_+$). For alternative of this, we obtain the following result.

Theorem 4 *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (1) and (4), respectively. Denote $\alpha_2 = \|AX_+^{-1}\|$ where X_+ is the extreme solution of Eq. (1),*

$$\begin{aligned}b_2 &= 1 - \alpha_2^2 + \|X_+\|\|X_+^{-1}\|^2\|\Delta Q\|\,, \\ c_2 &= \|\Delta Q\| + 2\alpha_2\|\Delta A\| + \|X_+^{-1}\|\|\Delta A\|^2\,.\end{aligned}$$

If

$$\alpha_2 < 1 \quad \text{and} \quad 2\|\Delta A\| + \|\Delta Q\| \leq \frac{(1 - \alpha_2)^2}{\|X_+\|\|X_+^{-1}\|^2}\,, \tag{16}$$

then $D_2 = b_2^2 - 4c_2\|X_+\|\|X_+^{-1}\|^2 \geq 0$ and

$$\|\Delta X_+\| \;\leq\; \|X_+\|\frac{b_2 - \sqrt{D_2}}{2} \;=:\; S_{err}^{P_2'}\,. \tag{17}$$

Proof: The proof is like to the proof of Theorem 3. □

3 Numerical examples

We consider some numerical examples for illustration of the theoretical results.

Example 1 [1, Example 2] *Consider Eq. (1) with coefficient matrices*

$$A = \begin{pmatrix} -0.4326 & -1.1465 \\ -1.6665 & 1.1909 \\ 0.1253 & 1.1892 \\ 0.2877 & -0.0376 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.1376 & 0.0656 \\ 0.0656 & 0.5616 \end{pmatrix}$$

and an approximation of the extreme solution

$$X_+ \approx \begin{pmatrix} 1.1572575 & 0.01971555 \\ 0.01971555 & 3.3569583 \end{pmatrix},$$

and perturbations on the matrices A and Q

$$\Delta A = 10^{-j} \begin{pmatrix} 1 & 0.6 \\ 0.2 & 0.4 \\ 0.8 & 0.4 \\ 0.6 & 0.1 \end{pmatrix}, \quad \Delta Q = 10^{-j} \begin{pmatrix} 0.4 & 0.7 \\ 0.7 & 0.3 \end{pmatrix},$$

respectively.

The solution $\tilde{X}_+$ to the perturbed equation we compute iteratively by formula

$$X_{k+1} = Q + A^* \widehat{X_k^{-1}} A, \quad X_0 = Q, \tag{18}$$

as $\tilde{X}_+ \approx \tilde{X}_{200}$.

We remain that for Example 1 we have $\|X_+^{-1}\|\|A\| \approx 1.9443 > 1$. Hence, Theorem 1 is not applicable. But $\|\widehat{PX_+^{-1}AP^{-1}}\| \approx 0.9621$ with $P = P_Q := \sqrt{Q} + 2\sqrt[4]{Q}$. Moreover for $P = P_1 := \sqrt{X_+}$ and $P = P_2 := X_+$ we have $\|\widehat{\sqrt{X_+^{-1}}A\sqrt{X_+^{-1}}}\| \approx 0.9472$ and $\|AX_+^{-1}\| \approx 0.5745$, respectively.

In Table 1 the perturbation estimates S_{err}^P (8) with different matrices P (P_Q, P_1, P_2), $S_{err}^{P_1'}$ (11) and $S_{err}^{P_2'}$ (17) for Example 1 with different values of j are given. The notations are the following. The sign "*" means that the inequalities $\|A\|^2\|X_+^{-1}\|^2 < 1$ and $\alpha_p < 1$ from (5) and (7) are not satisfied. In addition, the sign "**" means that these inequalities are satisfied and the second inequalities are not valid.

Example 2 ([13, Example 4.1]) *Consider Eq. (1) with $m = 2$, $n = 4$ and matrices A and Q as follows:*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad Q := X_+ - A^* \widehat{X_+^{-1}} A,$$

where

$$A_1 = \frac{1}{100} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 22.5 & 12 & 2 \\ 2 & 9 & 7 & 3 \\ 12 & 1 & 2 & 19 \end{pmatrix}, \quad A_2 = \frac{2\sqrt{3}}{45} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad X_+ = E + 1.5I$$

and E is the 4×4 matrix with all entries equal to 1.

Table 1: Numerical results of Example 1.

	$j=3$	$j=4$	$j=5$	$j=6$
$\|\Delta X_+\|$	$3.0137e-03$	$3.0193e-04$	$3.0159e-05$	$2.9752e-06$
S_{err}	$*$	$*$	$*$	$*$
$S_{err}^{P_Q}$	$**$	$1.1394e-02$	$1.0814e-03$	$1.0763e-04$
$S_{err}^{P_1}$	$**$	$9.6401e-03$	$9.4010e-04$	$9.3783e-05$
$S_{err}^{P_1'}$	$**$	$1.2038e-02$	$1.1667e-03$	$1.1632e-04$
$S_{err}^{P_2}$	$2.5665e-02$	$2.5396e-03$	$2.5370e-04$	$2.5367e-05$
$S_{err}^{P_2'}$	$3.7084e-02$	$3.6638e-03$	$3.6595e-04$	$3.6590e-05$

Consider perturbation on the matrices A and Q:

$$\Delta A = 10^{-2j} \begin{pmatrix} \frac{C_1+C_1^*}{\|C_1+C_1^*\|} \\ \frac{C_2+C_2^*}{\|C_2+C_2^*\|} \end{pmatrix}, \quad \Delta Q := \tilde{X}_+ - \tilde{A}^* \widehat{\tilde{X}_+^{-1}} \tilde{A} - Q,$$

with $\tilde{X}_+ = X_+ + 10^{-2j}(I-E)$, C_1, C_2 are random matrices generated by MATLAB function **randn**.

For Example 2 we have

$$\|X_+^{-1}\|\|A\| \approx 0.2119, \quad \|\widehat{X_+^{-1}A}\| \approx 0.1667, \quad \|\widehat{P_Q X_+^{-1} A P_Q}\| \approx 0.1667,$$

$$\|\widehat{\sqrt{X_+^{-1}} A \sqrt{X_+^{-1}}}\| \approx 0.1669, \quad \|AX_+^{-1}\| \approx 0.1095.$$

Therefore, Theorems 1 and 2 with $P = I$, $P = P_Q := \sqrt{Q} + 2\sqrt[4]{Q}$, $P = P_1 := \sqrt{X_+}$ and $P = P_2 := X_+$ and the theorems 3, and 4 are applicable.

In Table 2 the perturbation estimates S_{err} (6), S_{err}^P (8) with different matrices P (I, P_Q, P_1, P_2), $S_{err}^{P_1'}$ (11) and $S_{err}^{P_2'}$ (17) for Example 2 with different values of j are given.

Table 2: Numerical results of Example 2.

j	$j=2$	$j=3$	$j=4$	$j=5$
$\|\Delta X_+\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
S_{err}	$3.8082e-04$	$3.8893e-06$	$3.8605e-08$	$3.8588e-10$
S_{err}^I	$3.6109e-04$	$3.6899e-06$	$3.6628e-08$	$3.6604e-10$
$S_{err}^{P_Q}$	$3.6458e-04$	$3.6978e-06$	$3.6778e-08$	$3.6753e-10$
$S_{err}^{P_1}$	$4.4270e-04$	$4.5229e-06$	$4.5340e-08$	$4.6955e-10$
$S_{err}^{P_1'}$	$1.3242e-03$	$1.3532e-05$	$1.3433e-07$	$1.3424e-09$
$S_{err}^{P_2}$	$1.4957e-03$	$1.5630e-05$	$1.5450e-07$	$1.6050e-09$
$S_{err}^{P_2'}$	$4.5590e-03$	$4.6622e-05$	$4.6281e-07$	$4.6238e-09$

Example 3 *Consider Example 2 with* $A_1 = \kappa A_2^T$ *and* $X_+ = \lambda E + 2.5I$.

Next, in Table 3 the different norms are presented. These norms are important in order to the corresponding estimates are valid.

Table 3: Example 3 with different κ and λ.

	$\kappa = 10$ $\lambda = 0.1$	$\kappa = 12$ $\lambda = 0.1$	$\kappa = 13$ $\lambda = 0.1$	$\kappa = 11$ $\lambda = 0.91$
$\|X_+^{-1}\|\|A\|$	0.7903	0.9478	1.0265	0.8690
$\|\widehat{X_+^{-1}A}\|$	0.7870	0.9439	1.0224	0.8596
$\|\widehat{P_Q X_+^{-1}A P_Q}\|$	0.9246	1.3889	2.1834	0.7895
$\|\widehat{\sqrt{X_+^{-1}}A\sqrt{X_+^{-1}}}\|$	0.7489	0.8981	0.9728	0.6590
$\|AX_+^{-1}\|$	0.2846	0.3414	0.3697	0.2309

In the tables 4, 5 and 6 the perturbation estimates S_{err} (6), S_{err}^P (8) with different matrices P (I, P_Q, P_1, P_2), $S_{err}^{P_1'}$ (11) and $S_{err}^{P_2'}$ (17) for Example 3 with different values of j, κ and λ are given.

Table 4: Numerical results of Example 3 with $\kappa = 10$ and $\lambda = 0.1$.

j	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\|\Delta X_+\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
S_{err}	0.0013	$1.2940e-05$	$1.3016e-07$	$1.3809e-09$
S_{err}^I	0.0013	$1.2738e-05$	$1.2813e-07$	$1.3596e-09$
$S_{err}^{P_Q}$	0.0044	$4.4177e-05$	$4.4682e-07$	$4.8199e-09$
$S_{err}^{P_1}$	0.0011	$1.0837e-05$	$1.0906e-07$	$1.1587e-09$
$S_{err}^{P_1'}$	0.0012	$1.2527e-05$	$1.2601e-07$	$1.3390e-09$
$S_{err}^{P_2}$	$3.7772e-04$	$3.7749e-06$	$3.8123e-08$	$4.1385e-10$
$S_{err}^{P_2'}$	$5.0221e-04$	$5.0423e-06$	$5.0763e-08$	$5.5267e-10$

Usually, when the conditions of both theorems (Theorem 1 and Theorem 2) are satisfied, the estimate S_{err}^P derived in Theorem 2 is sharper than the estimate S_{err} given in Theorem 1.

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Table 5: Numerical results of Example 3 with $\kappa = 13$ and $\lambda = 0.1$.

j	$j=2$	$j=3$	$j=4$	$j=5$
$\|\Delta X_+\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
S_{err}	$*$	$*$	$*$	$*$
S_{err}^I	$*$	$*$	$*$	$*$
$S_{err}^{P_Q}$	$*$	$*$	$*$	$*$
$S_{err}^{P_1}$	0.0109	$9.8822e-05$	$1.0138e-06$	$1.0388e-08$
$S_{err}^{P_1'}$	0.0127	$1.1352e-04$	$1.1675e-06$	$1.1978e-08$
$S_{err}^{P_2}$	$4.3622e-04$	$4.2602e-06$	$4.3746e-08$	$4.5423e-10$
$S_{err}^{P_2'}$	$5.7776e-04$	$5.6026e-06$	$5.7782e-08$	$6.0210e-10$

Table 6: Numerical results of Example 3 with $\kappa = 11$ and $\lambda = 0.91$.

j	$j=2$	$j=3$	$j=4$	$j=5$
$\|\Delta X_+\|$	$3.0000e-04$	$3.0000e-06$	$3.0000e-08$	$3.0000e-10$
S_{err}	0.0020	$1.8875e-05$	$1.9722e-07$	$2.0909e-09$
S_{err}^I	0.0018	$1.7599e-05$	$1.8392e-07$	$1.9506e-09$
$S_{err}^{P_Q}$	0.0014	$1.4335e-05$	$1.4483e-07$	$1.4103e-09$
$S_{err}^{P_1}$	$9.5749e-04$	$9.8548e-06$	$9.7773e-08$	$9.4251e-10$
$S_{err}^{P_1'}$	0.0018	$1.7523e-05$	$1.8386e-07$	$1.9645e-09$
$S_{err}^{P_2}$	0.0010	$1.0279e-05$	$1.0403e-07$	$9.6852e-10$
$S_{err}^{P_2'}$	0.0019	$1.8134e-05$	$1.9285e-07$	$2.1128e-09$

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