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Numerical Solvers for the Stabilizing Solution to Riccati Type Equations Arising in Game Positive Models

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Abstract. We consider the linear quadratic differential games for positive linear systems with the feedback information structure in the general case. Recently, several iterative methods to obtain the stabilizing solution of a corresponding set of Riccati equations are described in the literature - the Newton method and its accelerated modification and the Lyapunov type iterative methods, which are applied to the game with two players. Here we extend the Lyapunov type iterative methods to compute the Nash equilibrium point to an N-player game. The sufficient conditions for convergence of the proposed methods are derived. The performances of the proposed algorithm are illustrated on some numerical examples.

Key Words: feedback Nash equilibrium, generalized Riccati equation, stabilizing solution, positive system, nonnegative solution.

1 Introduction

Many situations in economics, management, industry are characterized by decision makers (players) [3, 7]. Moreover, the linear quadratic differential games for positive systems have attracted considerable research interest [1, 2, 4]. We consider the linear quadratic differential games for positive systems, i.e. it is a special class of dynamic games, where the performance index is modeled by a quadratic function and the process is described by a linear differential equation with a positive solution.

Theory and algorithms based on the Nash feedback equilibria approach for finding the Nash equilibria are given in [5]. Based on the established Newton method in [2] we have extended the Newton method [12] to compute the Nash feedback equilibrium for linear quadratic differential games for positive systems with N players. In our previous papers [9, 10] we have considered Lyapunov type iterative methods to compute the Nash feedback equilibrium for linear quadratic differential games for positive systems with two players.

The infinite time horizon nonzero-sum linear quadratic (LQ) differential games of stochastic systems governed by Itôs equation with state and control-dependent noise is discussed in [13]. Authors have been presented necessary and sufficient conditions for the existence of the Nash strategy by means of four coupled stochastic algebraic Riccati equations. The problem of stochastic Nash differential games of Markov jump linear systems governed by Itô-type equation is investigated in [14]. Authors have been derived a necessary and sufficient condition for the existence of the Nash strategy by means of a set of cross coupled stochastic algebraic Riccati equations.

In this paper we extend the Lyapunov type iterative methods [9, 10] and apply them for finding the Nash feedback equilibrium for linear quadratic differential games on positive systems with N players. Since the the Nash feedback equilibrium is presented via the stabilizing solution of the associated coupled Riccati equations, we develop our approach to compute the stabilizing solution of a set of coupled set of Riccati equations.

Let us to introduce the game. The objective of player i $(i = 1, \dots, N)$ is defined as maximization of the own cost function, the latter being a quadratic functional J_i defined as follows

$$J_i(F_1, \dots, F_N, x_0) = \int_0^\infty x^T \left(Q_i + \sum_{j=1}^{N} F_j^T R_{ij} F_j \right) x \, dt \,, \tag{1}$$

where Q_i and R_{ij} are symmetric matrices with $Q_i \in \mathbb{R}^{n\times n}$ and $R_{ij} \in \mathbb{R}^{m_j \times m_j}$ and $i, j = 1, \dots, N$. The following additional requirements on the matrices are imposed:

(a) the matrices Q_i and $R_{ij}, (i \neq j)$ are symmetric and nonnegative;

(b) the matrix R_{ii} is negative definite and R_{ii}^{-1} is nonpositive, $i = 1, \dots, N$;

(c) the matrices $F_1, \dots, F_N$ belong to the set $\mathcal{F}$ of matrices:

$$\mathcal{F} = \{\mathbf{F} = (F_1, \dots, F_n) \text{ such that } F_j \in \mathbb{R}^{m_j \times n} \text{ and } A + \sum_{j=1}^{N} B_j F_j \text{ is asymptotically stable}\} \,.$$

The game is defined on the following dynamic system

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j \,, \quad x(0) = x_0 \tag{2}$$

where x is a state vector, $x_0 \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m_j}$ and u_j is a control vector, chosen by player $j, j = 1, \dots, N$. The controls u_j are of the type $u_j = F_j x$ and $F_j \in \mathbb{R}^{m_j \times n}$.

The above N-player infinite-horizon linear-quadratic differential game is applied to a positive system defined by (2). For this purpose, we introduce a definition as well as some facts and notations for nonnegative matrices and positive systems with the text that follows.

Definition 1 *The system (2) is said to be positive if for all initial nonnegative x_0 and for nonnegative controls $u_j, j = 1 \dots, N$, the state trajectory $x(t)$ takes only nonnegative values.*

Let us introduce some notations we are used in the paper. $\mathcal{R}^{n\times s}$ stands for $n \times s$ real matrices. The inequality $X \succcurlyeq 0$ $(X \succ 0)$ means that all elements of the matrix (or vector) X are real nonnegative (positive) and we call the matrix X nonnegative (positive). For the matrices $A = (a_{ij}), B = (b_{ij})$, we write $A \succcurlyeq B$, $(A \succ B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ hold for all indexes i and j. The notation $\mathbf{X} \succcurlyeq \mathbf{Y}$ with $\mathbf{X} = (X_1, \dots, X_N)$ means that $X_i \succcurlyeq Y_i \,, i = 1, \dots, N$. A matrix A is called asymptotically stable (or Hurwitz) if the eigenvalues of A have a negative real part. A symmetric matrix A is called positive definite (semidefinite) matrix if all eigenvalues are positive (nonnegative). An $n \times n$ matrix A is called a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be presented as $A = \alpha I - N$ with N being a nonnegative matrix, and it is called a nonsingular M-matrix if $\alpha > \rho(N)$, where $\rho(N)$ is the spectral radius of N. In addition, a matrix is called nonnegative (nonpositive) if all of its entries are nonnegative (nonpositive).

We would further use the following well-known property of positive systems (see [6]).

Proposition 2 *The system (2) is positive if and only if $B_j, j = 1, \ldots, N$ are nonnegative matrices and the matrix $-A$ is a Z-matrix.*

To be specific, it is necessary to introduce the following equilibrium definition:

Definition 3 *An N-tuple of matrices $\mathbf{F}^* = (F_1^*, \ldots, F_N^*)$ is called a deterministic feedback Nash equilibrium on the positive system (2) if the following inequalities hold:*

$$J_i(F_1^*, \ldots, F_N^*, x_0) \geq J_i(F_1^*, \ldots, F_{i-1}^*, F_i, F_{i+1}^*, \ldots, F_N^*, x_0)\,, \quad i = 1, \ldots, N\,,$$

for all initial nonnegative states x_0, all $F_i \in \mathbb{R}^{m_i \times n}$ such that $(F_1^*, \ldots, F_{i-1}^*, F_i, F_{i+1}^*, \ldots, F_N^*) \in \mathcal{F}$ and nonnegative strategies $u_i = F_i x, (F_i \succcurlyeq 0), i = 1, \ldots, N$.

Then, the deterministic feedback Nash equilibrium strategy is $u_i^* = F_i^*\, x(t)$ for player i, where $i = 1, \ldots, N$. Moreover, the state $x(t)$ is a solution to the following equation:

$$\dot{x} = \left(A + \sum_{j=1}^{N} B_j\, F_j^* \right) x\,, \quad x(0) = x_0 \succcurlyeq 0\,, \quad x \in [0, \infty)\,.$$

Definition 3 suggests that every player wants to maximize their utility function $J_i(\mathbf{F}, x_0)$.

2 Iterative methods

A deterministic feedback Nash equilibrium exists if and only if there exist N real symmetric $n \times n$ solutions X_i^* to the following set of equations (i=1,..., N):

$$\begin{aligned} 0 = \mathcal{R}_i(\mathbf{X}) := & -A^T X_i - X_i\, A - Q_i + X_i\, S_i\, X_i \\ & + \textstyle\sum_{j \neq i} (X_i\, S_j\, X_j + X_j\, S_j\, X_i - X_j\, S_{ij}\, X_j)\,, \end{aligned} \tag{3}$$

with the matrix $A - \sum_{j=1}^{N} S_j\, X_j^*$ is asymptotically stable. Moreover, the N-tuple of feedback matrices $(F_1^*, \ldots, F_N^*)$ with $F_i^* = -R_{ii}^{-1} B_i^T\, X_i^*$ is a deterministic feedback Nash equilibrium and

$$J_i(F_1^*, \ldots, F_N^*, x_0) = x_0^T\, X_i^*\, x_0\,, \qquad i = 1, \ldots, N.$$

Here, the matrix coefficients S_i and S_{ij} are $S_i = B_i\, R_{ii}^{-1}\, B_i^T$; $S_{ij} = B_j\, R_{jj}^{-1}\, R_{ij}\, R_{jj}^{-1}\, B_j^T\,,\ i, j = 1, \ldots, N,\ i \neq j$. Note that $S_i \preccurlyeq 0, i = 1, \ldots, N\,,\ S_{ij} \succcurlyeq 0, i, j = 1, \ldots, N\,, i \neq j$ in this investigation.

The matrix Y is a stabilizing solution to (3) if the matrix $A - \sum_{j=1}^{N} S_j\, Y_j$ is asymptotically stable.

The set of Riccati equations (3) should be solved so as to find a deterministic feedback Nash equilibrium point. The latter system is equivalent to a system of $Nn(n+1)/2$ quadratic scalar equations in $Nn(n+1)/2$ real scalar unknowns. Hence, at most $(Nn(n+1)/2)^2$ different solutions exist and the stability condition for each of them should be verified [11].

The Newton method is given by the formula [8] ($i = 1, \ldots, N$):

$$-{A^{(k)}}^T X_i^{(k+1)} - X_i^{(k+1)} A^{(k)} + \sum_{j\neq i} \left[W_{ij}^{(k)} X_j^{(k+1)} + X_j^{(k+1)} {W_{ij}^{(k)}}^T \right] = Q_i^{(k)} \,, \qquad (4)$$

where

$$\begin{aligned} & A^{(k)} = A - \textstyle\sum_j S_j X_j^{(k)} \,, \quad W_{ij}^{(k)} = X_i^{(k)} S_j - X_j^{(k)} S_{ij} \,, \quad i,j = 1, \ldots, N \,, \\ & Q_i^{(k)} = Q_i + X_i^{(k)} S_i X_i^{(k)} + \textstyle\sum_{j\neq i} [\, X_i^{(k)} S_j X_j^{(k)} + X_j^{(k)} S_j X_i^{(k)} - X_j^{(k)} S_{ij} X_j^{(k)} \,] \,. \end{aligned} \qquad (5)$$

The set of matrix equations (4) is equivalent to the linear system:

$$L^{(k)} \; vec(X_1^{(k+1)}, \ldots, X_N^{(k+1)}) \; = \; vec(Q_1^{(k)}, \ldots, Q_N^{(k)}) \,,$$

where

$$L^{(k)} = \left(L_{ij}^{(k)} \right)_{i,j=1}^{N} = \begin{cases} L_{ii}^{(k)} = -I_n \otimes {A^{(k)}}^T - {A^{(k)}}^T \otimes I_n \\ L_{ij}^{(k)} = -I_n \otimes {W_{ij}^{(k)}}^T - {W_{ij}^{(k)}}^T \otimes I_n, \quad i \neq j \,. \end{cases} \qquad (6)$$

The convergence properties of iterative method (4)-(5) are established in Theorem 2.3 [8].

In addition, the accelerated Newton method is presented in the same paper:

$$-{A^{(k)}}^T X_i^{(k+1)} - X_i^{(k+1)} A^{(k)} = \tilde{Q}_i^{(k)}, \qquad i = 1, \ldots, N \,, \qquad (7)$$

where

$$\begin{aligned} \tilde{Q}_i^{(k)} \;\; = \;\; & Q_i^{(k)} - \textstyle\sum_{j<i} \left[W_{ij}^{(k)} X_j^{(k+1)} + X_j^{(k+1)} {W_{ij}^{(k)}}^T \right] \\ & - \textstyle\sum_{j>i} \left[W_{ij}^{(k)} X_j^{(k)} + X_j^{(k)} {W_{ij}^{(k)}}^T \right] \,. \end{aligned} \qquad (8)$$

In Theorem 2.5 [8] the convergence properties of the accelerated Newton method for a N-player differential game, where the information structure of each player is of a feedback patten are derived. In order to improve the Newton method we introduce the Lyapunov iterative process, where the sequences of Lyapunov algebraic equations are constructed.

We put $X_j^{(k)}$ instead of $X_j^{(k+1)}$ in the first equation of (4). We obtain a new iterative method named the Lyapunov method:

$$-{A^{(k)}}^T X_i^{(k+1)} - X_i^{(k+1)} A^{(k)} \; = \; \hat{Q}_i^{(k)}, \qquad i = 1, \ldots, N \,, \qquad (9)$$

where

$$\hat{Q}_i^{(k)} \; = \; Q_i^{(k)} + X_i^{(k)} S_i X_i^{(k)} + \textstyle\sum_{j\neq i} [\, X_j^{(k)} S_{ij} X_j^{(k)} \,] \,. \qquad (10)$$

In our investigation we exploit the fact that the following statements are equivalent for a Z-matrix (-A):

(a) $-A$ is a nonsingular M-matrix;
(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;
(c) A is asymptotically stable.

In addition, the following identity is true [8]:

$$\begin{aligned} \mathcal{R}_i(\mathbf{X}) \;=\; \mathcal{R}_i(\mathbf{Z},\mathbf{X}) \;:=\; & -A_{\mathbf{Z}}{}^T X_i - X_i A_{\mathbf{Z}} - Q_i - Z_i S_i Z_i \\ & +(X_i - Z_i)S_i(X_i - Z_i) \\ & + \textstyle\sum_{j\neq i} [(X_j - Z_j) S_j X_i + X_i S_j (X_j - Z_j)] - \textstyle\sum_{j\neq i} X_j S_{ij} X_j \,, \end{aligned} \tag{11}$$

where $A_{\mathbf{Z}} = A - \sum_j S_j Z_j \,,\ Z_i = Z_i^T,\ i = 1, \ldots, N.$

We extend the results derived for nonnegative matrices in Lemma 1.3 from [8] in the following statement:

Lemma 4 *For a matrix $A \in \mathcal{R}^{n\times n}$ such that (-A) is a Z-matrix, the following statements are equivalent:*
(i) $-A$ is a nonsingular M-matrix;
(ii) for any nonnegative symmetric $Q \in \mathcal{R}^{n\times n}$ the Lyapunov equation $-A^T X - XA = Q$ has a unique nonnegative solution X ;
(iii) there exists nonnegative $Q_0 = Q_0^T$ such that the equation $-A^T X - XA = Q_0$ has a nonnegative solution X_0 .

Proof: (i)→ (ii) Assume A is asymptotically stable and -A is a Z-matrix. Then $(-A^T)\otimes I_n + I_n \otimes (-A^T)$ is a nonsingular M-matrix and $vec(X) = [(-A^T)\otimes I_n + I_n \otimes (-A^T)]^{-1}\, vec(Q) \succ 0$ when Q is nonnegative. Thus (i) implies (ii).

(ii)→ (iii) The proof is obvious because (iii) is a special case of (ii).

(iii)→ (i) Let us consider $X_\varepsilon = X_0 + \varepsilon E_n$, where X_0 is the solution of $-A^T X - XA = Q$ and $E_n \in \mathcal{R}^{n\times n}$ is the matrix with all its elements equal with 1. Then $X_\varepsilon \succ 0$ for any $\varepsilon > 0$, because $X_0 \succcurlyeq 0$. One obtains that X_ε solves the Lyapunov equation $-A^T X_\varepsilon - X_\varepsilon A = Q_\varepsilon$, where $Q_\varepsilon = Q_0 - \varepsilon(A^T E_n + E_n A)$. Since $Q_0 \succ 0$ there exists $\varepsilon_0 > 0$ such that $Q_\varepsilon \succ 0$ for any $\varepsilon \in (0, \varepsilon_0]$. Further on, $[(-A^T) \otimes I_n + I_n \otimes (-A^T)]\, vec(X_\varepsilon) = vec(Q_\varepsilon) \succ 0$. Thus, $(-A^T) \otimes I_n + I_n \otimes (-A^T)$ is a nonsingular M-matrix and (i) holds.

□

Remark 5 *Often in the applications one may show that there exists a $Q_0 \succ 0$ such that a Lyapunov equation $-A^T X - XA = Q_0$ has a nonnegative solution X_0 without being able to prove that $X_0 \succ 0$. The implication (iii) → (i) from the above lemma shows that this allows us to conclude that $(-A)$ is a nonsingular M-matrix, which is equivalent to the fact that A is a Hurwitz matrix. The condition $X_0 \succ 0$ cannot be relaxed to $Q_0 \succcurlyeq 0$. This can be seen from the following example: $A = diag[-1, 1]\,, Q_0 = diag[2, 0]$. The equation $-A^T X - XA = Q_0$ has the solution $X_0 = diag[1,\ 0] \succcurlyeq 0$ but A is not a Hurwitz matrix. That is way we introduce the assumption of $\mathcal{R}_i(\hat{X}_1, \ldots, \hat{X}_N) \succ 0$ in the next theorem.*

Lemma 6 *If there exist the symmetric nonnegative matrix $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_N)$ and a number $i_0\,, 1 \le i_0 \le N$ such that $\mathcal{R}_{i_0}(\hat{\mathbf{X}}) \succ 0$ then $(-A)$ is a nonsingular M-matrix.*

Proof: Consider equation (3) with X_i replaced by $\hat{X}_i$:

$$-A^T \hat{X}_i - \hat{X}_i A = \hat{Q}_i$$

where $\hat{Q}_i = \mathcal{R}_i(\hat{\mathbf{X}}) + Q_i - \hat{X}_i S_i \hat{X}_i - \sum_{j \neq i} [\hat{X}_i S_j \hat{X}_j + \hat{X}_j S_j \hat{X}_i - \hat{X}_j S_{ij} \hat{X}_j]$. Since $\mathcal{R}_{i_0}(\hat{\mathbf{X}}) \succ 0$ we have $Q_{i_0} \succ 0$. Moreover, applying the implication (iii) $\to$ (i) from Lemma 4 in the case of the equation $-A^T \hat{X}_{i_0} - \hat{X}_{i_0} A = \hat{Q}_{i_0}$ we obtain that $(-A)$ is a nonsingular M-matrix. □

The convergence properties of the Lyapunov iteration (9)-(10) are established in the following theorem:

Theorem 7 *Assume there exist symmetric nonnegative matrices* $\hat{X}_1, \dots, \hat{X}_N$ *and* $X_1^{(0)} = \dots = X_N^{(0)} = 0$ *and a number* $i_0 \, , 1 \leq i_0 \leq N$ *such that* $\mathcal{R}_{i_0}(\hat{X}_1, \dots, \hat{X}_N) \succ 0$. *Then, the matrix sequences* $\{X_1^{(k)}, \dots, X_N^{(k)}\}_{k=0}^{\infty}$ *defined by (9)-(10) satisfies:*

(i) $\hat{X}_i \succcurlyeq X_i^{(k+1)} \succcurlyeq X_i^{(k)}$ *and* $\mathcal{R}_i(X_1^{(k)}, \dots, X_N^{(k)}) \preccurlyeq 0$ *for* $i = 1, \dots, N \, , \; k = 0, 1, \dots;$

(ii) *The matrix* $(-A^{(k)})$ *is a nonsingular M-matrix for* $k = 0, 1, \dots;$

(iii) *The matrix sequences* $\{X_1^{(k)}, \dots, X_N^{(k)}\}_{k=0}^{\infty}$ *converge to the nonpositive solution* $\tilde{X}_1, \dots, \tilde{X}_N$ *to the set of Riccati equations (3) with* $\tilde{X}_i \preccurlyeq \hat{X}_i$ *and the matrix* $\tilde{A} = A - \sum_j S_j \tilde{X}_j$ *is asymptotically stable.*

Proof: Under assumption $\mathcal{R}_i(\hat{X}_1, \dots, \hat{X}_N) \succ 0$ and Lemma 6 we conclude that $(-A)$ is a nonsingular M-matrix. Using iteration (9) we construct the matrix sequences

$$X_1^{(1)}, \dots, X_N^{(1)}; X_1^{(2)}, \dots, X_N^{(2)}; \dots, X_1^{(r)}, \dots, X_N^{(r)} .$$

We will prove by induction the following statements for $r = 0, \dots$:

(A) $\mathcal{R}_i(X_1^{(r)}, \dots, X_N^{(r)}) \preccurlyeq 0, i = 1, \dots, N$ and the matrix $(-A^{(r)})$ is an M-matrix;

(B) $X_i^{(r+1)} \succcurlyeq X_i^{(r)} \, , \; i = 1, \dots, N;$

(C) $\hat{X}_i \succcurlyeq X_i^{(r+1)} \, , \; i = 1, \dots, N.$

Assume that $\mathcal{R}_i(X_1^{(k-1)}, \dots, X_N^{(k-1)}) \preccurlyeq 0$ and the matrix $(-A^{(k-1)})$ is a nonsingular M-matrix and $\hat{X}_i \succcurlyeq X_i^{(k)} \succcurlyeq X_i^{(k-1)} \, , i = 1, \dots, N$. We will prove the statements (A)-(B)-(C) for $r = k$.

First, we would prove $\mathcal{R}_i(X_1^{(k)}, \dots, X_N^{(k)}) \preccurlyeq 0, i = 1, \dots, N$ and $(-A^{(k)})$ is a nonsingular M-matrix. Secondly, we would compute $X_1^{(k+1)}, \dots, X_N^{(k+1)}$ as a unique solution of (9). Third, we would prove that $\hat{X}_i \succcurlyeq X_i^{(k+1)} \succcurlyeq X_i^{(k)} \, , i = 1, \dots, N$.

Using identity (11) for $\mathcal{R}_i(X_1^{(k)}, \dots, X_N^{(k)})$ we present

$$\begin{aligned}
&\mathcal{R}_i(X_1^{(k)}, \dots, X_N^{(k)}) = \mathcal{R}_i(X_1^{(k-1)}, \dots, X_N^{(k-1)}, X_1^{(k)}, \dots, X_N^{(k)}) \\
&= -{A_{k-1}}^T X_i^{(k)} - X_i^{(k)} A_{k-1} - Q_i - X_i^{(k-1)} S_i X_i^{(k-1)} \\
&\quad + (X_i^{(k)} - X_i^{(k-1)}) S_i (X_i^{(k)} - X_i^{(k-1)}) \\
&\quad + \textstyle\sum_{j \neq i} [(X_j^{(k)} - X_j^{(k-1)}) S_j X_i^{(k)} + X_i^{(k)} S_j (X_j^{(k)} - X_j^{(k-1)})] - \sum_{j \neq i} X_j^{(k)} S_{ij} X_j^{(k)} .
\end{aligned}$$

However, we know

$$-A^{(k-1)T} X_i^{(k)} - X_i^{(k)} A^{(k-1)} - Q_i - X_i^{(k-1)} S_i X_i^{(k-1)} = + \textstyle\sum_{j \neq i} X_j^{(k-1)} S_{ij} X_j^{(k-1)} .$$

We obtain

$$\mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)}) = -\sum_{j\neq i}\Big(X_j^{(k-1)}S_{ij}(X_j^{(k)}-X_j^{(k-1)}) + (X_j^{(k)}-X_j^{(k-1)})\,S_{ij}\,X_j^{(k)}\Big)$$
$$+(X_i^{(k)}-X_i^{(k-1)})S_i(X_i^{(k)}-X_i^{(k-1)})$$
$$+\sum_{j\neq i}[(X_j^{(k)}-X_j^{(k-1)})\,S_j\,X_i^{(k)} + X_i^{(k)}\,S_j(X_j^{(k)}-X_j^{(k-1)})]\,.$$

Since $S_i \preccurlyeq 0, i=1,\dots,N,\ S_{ij}\succcurlyeq 0,\ i,j=1,\dots,N,\ j\neq i$ and hence, together with $X_i^{(k)} \succcurlyeq X_i^{(k-1)} \succcurlyeq 0, i=1,\dots,N$ we infer that $\mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)}) \preccurlyeq 0\,, i=1,\dots,N$.

Next, we will prove that $-A^{(k)}$ is a nonsingular M-matrix. We consider the difference

$$\mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)}) - \mathcal{R}_i(\hat X_1,\dots,\hat X_N)$$
$$= \mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)}) - \mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)},\hat X_1,\dots,\hat X_N)$$

we derive

$$-{A^{(k)}}^T\,(X_i^{(k)}-\hat X_i) - (X_i^{(k)}-\hat X_i)\,A^{(k)} = T_i^{(k)}\,, \tag{12}$$

and

$$\begin{aligned} T_i^{(k)} \quad := \quad & \mathcal{R}_i(\mathbf{X}^{(k)}) - \mathcal{R}_i(\hat{\mathbf{X}}) + (\hat X_i - X_i^{(k)})S_i(\hat X_i - X_i^{(k)}) + (\hat X_i - X_i^{(k)})S_i(\hat X_i - X_i^{(k)}) \\ & -\sum_{j\neq i}^{N}[\hat X_j\,S_{ij}\,(\hat X_j - X_j^{(k)}) + (\hat X_j - X_j^{(k)})\,S_{ij}\,X_j^{(k)}] \\ & +\sum_{j\neq i}[(\hat X_j - X_j^{(k)})\,S_j\,\hat X_i + \hat X_i\,S_j(\hat X_j - X_j^{(k)})]\,. \end{aligned}$$

Since $\mathcal{R}_i(\hat X_1,\dots,\hat X_N) \succ 0, \mathcal{R}_i(X_1^{(k)},\dots,X_N^{(k)}) \preccurlyeq 0$ and $S_i \preccurlyeq 0, i=1,\dots,N$ and $\hat X_j\,S_{ij}\,(\hat X_j - X_j^{(k)}) \succcurlyeq 0,\ (\hat X_j - X_j^{(k)})\,S_{ij}\,X_j^{(k)} \succcurlyeq 0$, and hence, together with $\hat{\mathbf{X}} \succcurlyeq \mathbf{X}^{(k)} \succcurlyeq 0$ we infer that the right hand of the above identity is nonpositive, i.e. $T_i^{(k)} \prec 0, i=1,\dots,N$. Applying the implication (iii) $\to$ (i) from Lemma 4 in the case of equation (12) we conclude that the matrix $-A^{(k)}$ is a nonsingular M-matrix.

Thus, we can apply the recursive equation (9) to find the matrix $X_1^{(k+1)},\dots,X_N^{(k+1)}$. We will prove $\hat X_i \succcurlyeq X_i^{(k+1)}, i=1,\dots,N$.

Combining $\mathcal{R}_i(\hat{\mathbf{X}}) = \mathcal{R}_i(\mathbf{X}^{(k)},\hat{\mathbf{X}})$ with (9) we compute

$$\begin{aligned} -\mathcal{R}_i(\hat{\mathbf{X}}) \quad := \quad & -{A^{(k)}}^T\,(X_i^{(k+1)}-\hat X_i) - (X_i^{(k+1)}-\hat X_i)\,A^{(k)} \\ & -(\hat X_i - X_i^{(k)})S_i(\hat X_i - X_i^{(k)}) - \textstyle\sum_{j\neq i} X_j^{(k)}S_{ij}X_j^{(k)} \\ & -\textstyle\sum_{j\neq i}[(\hat X_j - X_j^{(k)})\,S_j\,\hat X_i + \hat X_i\,S_j(\hat X_j - X_j^{(k)})] + \sum_{j\neq i}\hat X_j\,S_{ij}\,\hat X_j\,, \end{aligned}$$

and thus

$$\begin{aligned} & -{A^{(k)}}^T\,(X_i^{(k+1)}-\hat X_i) - (X_i^{(k+1)}-\hat X_i)\,A^{(k)} \\ := \quad & -\mathcal{R}_i(\hat{\mathbf{X}}) + (\hat X_i - X_i^{(k)})S_i(\hat X_i - X_i^{(k)}) \\ & +\textstyle\sum_{j\neq i}[X_j^{(k)}S_{ij}X_j^{(k)} - \hat X_j\,S_{ij}\,\hat X_j \pm \hat X_j\,S_{ij}\,X_j^{(k)}] \\ & +\textstyle\sum_{j\neq i}[(\hat X_j - X_j^{(k)})\,S_j\,\hat X_i + \hat X_i\,S_j(\hat X_j - X_j^{(k)})] \end{aligned}$$

Moreover,

$$\begin{array}{rl} & -A^{(k)^T}\,(X_i^{(k+1)} - \hat{X}_i) - (X_i^{(k+1)} - \hat{X}_i)\,A^{(k)} \\ := & -\mathcal{R}_i(\hat{\mathbf{X}}) + (\hat{X}_i - X_i^{(k)})S_i(\hat{X}_i - X_i^{(k)}) \\ & -\sum_{j\neq i}[(\hat{X}_j - X_j^{(k)})\,S_{ij}\,X_j^{(k)} + \hat{X}_j\,S_{ij}\,(\hat{X}_j - X_j^{(k)})] \\ & +\sum_{j\neq i}[(\hat{X}_j - X_j^{(k)})\,S_j\,\hat{X}_i + \hat{X}_i\,S_j(\hat{X}_j - X_j^{(k)})]\,. \end{array}$$

Now let us analyze the last set of matrix equations. The matrix $-A^{(k)}$ is a nonsingular M-matrix. The right-hand side of each equation is nonpositive. Thus $X_i^{(k+1)} - \hat{X}_i \succcurlyeq 0\,, i = 1, \ldots, N$ and $\hat{\mathbf{X}} \succcurlyeq \mathbf{X}^{(k+1)}$.

For proving $\mathbf{X}^{(k+1)} \succcurlyeq \mathbf{X}^{(k)}$ we combine $\mathcal{R}_i(\mathbf{X}^{(s)})$ with iteration (9). We infer

$$-\mathcal{R}_i(\mathbf{X}^{(k)}) = -A_{\mathbf{X}^{(k)}}{}^T\,(X_i^{(k+1)} - X_i^{(k)}) - (X_i^{(k+1)} - X_i^{(k)})\,A_{\mathbf{X}^{(k)}}\,.$$

Since $\mathcal{R}_i(\mathbf{X}^{(k)})$ is a nonpositive matrix and $-A^{(k)}$ is a nonsingular M-matrix we obtain $X_i^{(k)} - X_i^{(k+1)} \preccurlyeq 0, i = 1, 2$. Thus $\mathbf{X}^{(k+1)} \succcurlyeq \mathbf{X}^{(k)}$. Hence, the induction process has been completed.

Thus the matrix sequences $\{X_1^{(k)}, \ldots, X_N^{(k)}\}_{k=0}^{\infty}$ are monotonically increasing and bounded above by $(\hat{X}_1, \ldots, \hat{X}_N)$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(X_1^{(k)}, \ldots, X_N^{(k)}) = (\tilde{X}_1, \ldots, \tilde{X}_N)$. By taking the limits in (9) it follows that $(\tilde{X}_1, \ldots, \tilde{X}_N)$ is a solution of $\mathcal{R}_i(\mathbf{X}) = 0, i = 1, \ldots, N$ with the property $(\tilde{X}_1, \ldots, \tilde{X}_N) \preccurlyeq (\hat{X}_1, \ldots, \hat{X}_N)$ and $(-\tilde{A})$ is an M-matrix and therefore $\tilde{A}$ is asymptotically stable. □

In order to improve the Lyapunov method (9)-(10) we change $\hat{Q}_i^{(k)}\,, i = 2, \ldots, N$ as follows

$$\begin{array}{l} \tilde{\hat{Q}}_i^{(k)} = Q_i + X_i^{(k)} S_i X_i^{(k)} + \sum_{j<i} X_j^{(k+1)} S_{ij} X_j^{(k+1)} \\ +\sum_{j>i} X_j^{(k)} S_{ij} X_j^{(k)}\,, \qquad i = 1, \ldots, N\,. \end{array} \tag{13}$$

The convergence properties of the accelerated Lyapunov iteration (9)-(13) are established in the following theorem:

Theorem 8 *Assume there exist symmetric nonnegative matrices* $\hat{X}_1, \ldots, \hat{X}_N$ *and* $X_1^{(0)} = \ldots = X_N^{(0)} = 0$ *and a number* $i_0\,, 1 \le i_0 \le N$ *such that* $\mathcal{R}_{i_0}(\hat{X}_1, \ldots, \hat{X}_N) \succ 0$. *Then, the matrix sequences* $\{X_1^{(k)}, \ldots, X_N^{(k)}\}_{k=0}^{\infty}$ *defined by (9)-(13) satisfies:*

(i) $\hat{X}_i \succcurlyeq X_i^{(k+1)} \succcurlyeq X_i^{(k)}$ *and* $\mathcal{R}_i(X_1^{(k)}, \ldots, X_N^{(k)}) \preccurlyeq 0$ *for* $i = 1, \ldots, N\,,\ k = 0, 1, \ldots$*;*

(ii) *The matrix* $-A^{(k)}$ *is an M-matrix for* $k = 0, 1, \ldots$*;*

(iii) *The matrix sequences* $\{X_1^{(k)}, \ldots, X_N^{(k)}\}_{k=0}^{\infty}$ *converge to the nonpositive solution* $\tilde{X}_1, \ldots, \tilde{X}_N$ *to the set of Riccati equations (3) with* $\tilde{X}_i \preccurlyeq \hat{X}_i$ *and the matrix* $\tilde{A}$ *is asymptotically stable.*

Proof: The theorem is proved following the proof of Theorem 7.

Note that the accelerated Lyapunov method preserves the convergence properties of the Lyapunov method (9)-(10).

Table 1: Example 1. Results from 100 runs for each value of n.

	NI: (4)			ANI: (7)-(8)			LI: (9)-(10)			ALI: (9)-(13)		
n	It	$avIt$	CPU	It	$avIt_L$	CPU	It	$avIt$	CPU	It	$avIt$	CPU
10	5	5	2.0s	12	10.4	0.92s	12	10.4	1.0s	11	4.1	0.8s
11	5	5	3.0s	13	10.6	1.1s	13	10.6	1.0s	11	9.4	0.9s
12	5	5	4.3s	14	10.8	1.3s	14	10.8	0.9s	12	9.6	1.0s
13	5	5	6.0s	14	11.1	1.3s	14	11.1	1.2s	12	10.0	1.1s
14	5	5	8.8s	14	11.5	1.4s	14	11.4	1.2s	13	10.2	1.1s
15	5	5	14.4s	15	11.7	1.7s	15	11.7	1.6s	13	10.5	1.4s

3 Numerical experiments

We carry out some numerical experiments for computing the stabilizing solution to the set of generalized Riccati equations (3). The Newton iteration (4), the accelerated Newton iteration (ANI) (7)-(8), and Lyapunov iteration (LI) (9)-(10) and accelerated Lyapunov iteration (ALI) (9)-(13) are applied and compared on some examples.

Example 1. We consider a three-player game where the matrix coefficients: A, B_i, Q_i and R_{ij} for $i, j = 1, 2, 3$ are the following. We define them using the Matlab description.

A=abs(randn(n,n))/10 -3*eye(n,n);
B_1= zeros(n,1); B_1(1,1)=5; B_1(3,1)=2; B_1(n,1)=4;
B_2=full(abs(sprandn(n,4,0.8))/10);
B_3=full(abs(sprandn(n,3,0.8))/10);
R_{11} = -90;
R_{22} = [-400 0 0 -10; 0 -100 0 0; 0 0 -200 0; -10 0 0 -400];
R_{33} = [-800 0 0; 0 -900 -50; 0 -50 -600];
R_{21} = 200;
R_{31} = 200;
R_{12} = [40 0 0 0; 0 200 0 0; 0 0 500 0;0 0 0 30];
R_{13}= [120 0 0; 0 75 0; 0 0 140];
R_{23}= [220 0 0; 0 180 0; 0 0 190];
R_{32} = [100 0 0 0; 0 250 0 0; 0 0 240 0;0 0 0 300];
Q_1=4.5*eye(n,n); Q_1(1,n)=sqrt(n/2); Q_1(n,1)=sqrt(n/2);
Q_2=3.75*eye(n,n); Q_2(1,n)=4.5; Q_2(n,1)=4.5;
Q_3=2.85*eye(n,n); Q_3(1,n)=1/sqrt(n/2); Q_3(n,1)=1/sqrt(n/2);

Example 2. The matrix coefficients are:
B_1=full(abs(sprandn(n,4,0.7))/10);
B_2=full(abs(sprandn(n,3,0.7))/10);
B_3=full(abs(sprandn(n,3,0.8))/10);,
R_{11} = [-400 0 0 -40; 0 -150 0 0; 0 0 -300 0; -40 0 0 -300];
R_{22} = [-90 0 0; 0 -120 -5; 0 -5 -120];
R_{33} = [-800 0 0; 0 -900 -50; 0 -50 -600];
R_{21} = [100 88 0 99; 88 250 190 0; 0 190 240 130; 99 0 130 300];
R_{31} = 200;

Table 2: Example 2. Test 1. Results from 100 runs for each value of n.

	NI: (4)			ANI: (7)-(8)			LI: (9)-(10)			ALI: (9)-(13)		
n	It	$avIt$	CPU	It	$avIt_L$	CPU	It	$avIt$	CPU	It	$avIt$	CPU
10	4	3.9	2.8s	13	9.2	1.5s	12	9.1	1.3s	11	8.0	1.1s
11	4	4	4.4s	13	9.7	1.7s	13	9.6	1.6s	11	8.5	1.2s
12	4	4	5.6s	15	10.6	1.8s	15	10.5	1.7s	13	9.1	1.4s
13	5	4	8.4s	17	11.4	2.4s	17	11.3	2.0s	14	9.8	1.8s
14	5	4	11.7s	18	12.6	2.6s	18	12.5	2.4s	15	10.6	2.1s
15	5	4	17.4s	24	14.9	3.5s	24	14.7	3.0s	18	12.1	2.5s
16	5	4.5	18.0s	30	18.1	2.7s	30	18.0	2.3s	22	14.5	1.9s

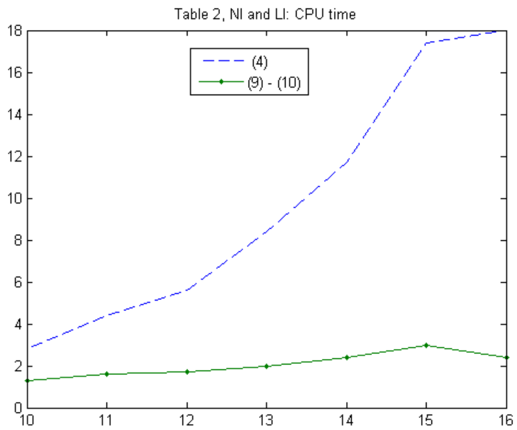

Figure 1: Comparison for CPU time

R_{12} = [220 190 190; 190 180 22; 190 22 190];
R_{13} = [120 0 0; 0 75 0; 0 0 140];
R_{23} = [220 0 0; 0 180 0; 0 0 190];
R_{32} = [100 0 0; 0 250 0; 0 0 240];
Q_1=4.5*eye(n,n); Q_1(1,n)=sqrt(n/2); Q_1(n,1)=sqrt(n/2);
Q_2=3.75*eye(n,n); Q_2(1,n)=4.5; Q_2(n,1)=4.5;
Q_3=2.85*eye(n,n); Q_3(1,n)=1/sqrt(n/2); Q_3(n,1)=1/sqrt(n/2);
Test 1: A=(abs(rand(n,n))/2-6*eye(n,n))/10;

On the basis of the experiments, performed for $n = 15$, the following conclusions might be outlined. The Newton iteration (4) requires 4 iteration steps (see Table 2) while finding the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_N$. Yet, the average number of iteration steps is 14.9 and they executed from the accelerated Newton method (7) so as to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_{ANI}$. The CPU time is 1.4s and 0.5s for executing the Newton and the accelerated Newton iteration respectively for 100 runs. Moreover, the results for Lyapunov type methods are: the average number of iteration steps are 14.7 and 12.1, and CPU time is 3s and 2.5s for executing the Lyapunov

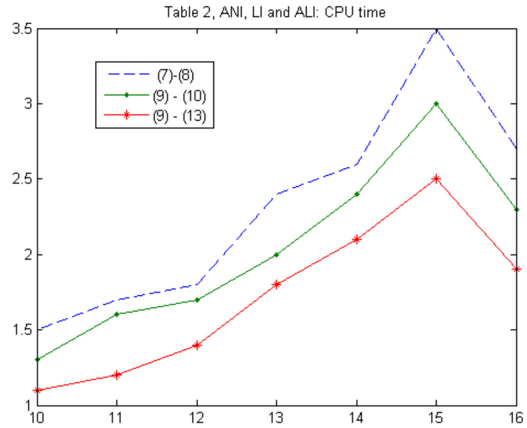

Figure 2: Comparison for CPU time

method and the accelerated Lyapunov method, respectively for 100 runs.

4 Conclusion

We have studied four iterative processes for finding the stabilizing solution to a set of generalized Riccati equations (3). The convergence properties of the Lyapunov type methods to compute a stabilizing solution to (3) are derived. Numerical experiments are carried out and the obtained results are used for comparison purposes. Thus, the following conclusions might be outlined. On one hand, the effectiveness of the proposed new iterative methods (9)-(10) and (9)-(13) is confirmed. On the other hand the Lyapunov type methods, based on the solution the Lyapunov equations at each iteration step, is found to be faster than the Newton iterations.

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