

On Accelerated Inexact Newton Method for a Nonlinear Matrix Equation *

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Abstract. In this paper we consider an accelerated inexact Newton method for finding the positive definite solution of the nonlinear matrix equation $X - A^*X^{-1}A = Q$ introduced in [1]. This method is based on Woodbury formula. The same technique has been previously used by Fang et al. [2] for solving equations $X - A^*\bar{X}^{-1}A = I$ and $X - A^*X^{-1}A = I$ by accelerated fixed-point iteration, inverse-free variant, Cyclic reduction algorithm and Newton's iteration. Here we compare the above methods and the considered accelerated inexact Newton method by numerical experiments.

Key words: nonlinear matrix equations, positive definite solutions, iterative methods.

1 Introduction

We consider an accelerated inexact Newton method for solving the nonlinear matrix equation

$$X - A^*X^{-1}A = Q, \quad (1)$$

where, Q is an $n \times n$ Hermitian positive definite matrix, A is an $n \times n$ matrix, and A^* denotes the conjugate transpose of A .

Eq. (1) and its Hermitian solutions have studied by Ferrante and Levy [3]. They by using Woodbury matrix identity formula (or Sherman–Morrison–Woodbury formula) have obtained its relationship with following algebraic Riccati equation

$$X = Q + FXF^* - FX(X + R)^{-1}XF^*$$

in case of $F = A^*A^{-1}$ and $R = AQ^{-1}A^*$. They prove that Eq. (1) always has a unique positive definite solution (PDS) which is the largest Hermitian solution and give an iterative

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method (Fixed-point iteration) for finding this solution. Eq. (1) can be reduced to the equation $X - A_1^* X^{-1} A_1 = I$ with right-hand side I (I is identity matrix) [3].

Another matrix equation intensively studied in last three decades is

$$Y + B^* Y^{-1} B = P. \quad (2)$$

Eq. (2) have been introduced by Anderson et al. [4]. Engwerda et al [5] and Engwerda [6] have investigated the theoretical properties of Eq. (2) as necessary and sufficient conditions for the existence of a PDS. They also have given conditions for a largest and smallest positive definite solutions (PDSs) and iterative methods (Fixed-point iterations) for finding them. Moreover it is proven the connection of (2) with an algebraic Riccati equation.

Besides the fixed-point iterations presented in [3, 5, 6], some authors have proposed other iterative methods for finding PDSs. Zhan [7] have given sufficient condition for linear convergence rate of the fixed-point iteration (FPI) for Eq. (2) and have proposed an inverse-free variant of the FPI. Guo and Lancaster [8] have proposed second inverse-free variant of the FPI for Eq. (2) and Newton's method for both equations (1) and (2). Meini [9] has studied a cyclic reduction method for finding the largest and the smallest solutions of the equations (1) and (2), respectively.

Motivated by connectivity of Eq. (1) and Eq. (2) with an algebraic Riccati equation, we obtained in [1] relationship between Eq. (1) and Eq. (2) by using Woodbury matrix identity formula (or Sherman–Morrison–Woodbury formula). Moreover, in [1] we considered inexact Newton's methods for finding the largest solutions of the equations (1) and (2), respectively.

Fang et al. [2] by using same technique has obtained a relationship between equation $X - A^* \bar{X}^{-1} A = I$ ($X - A^* X^{-1} A = I$) and Eq. (2). Moreover, they proved that: if $X_+ > 0$ is the unique positive definite solution of $X - A^* \bar{X}^{-1} A = I$ ($X - A^* X^{-1} A = I$), then $X_+ = Y_L - \bar{A} \bar{A}^*$ ($X_+ = Y_L - A A^*$), where Y_L is the maximal positive definite solution of Eq. (2), where $B = \bar{A} A$ ($B = A^2$) and $P = I + A^* A + \bar{A} \bar{A}^*$ ($P = I + A^* A + A A^*$) (see Theorem 3.3., (i) and Theorem 4.1. (i) in [2]).

The paper is organized as follows: In Section 2, we present some known iterative methods. Present relationship between the equations (1) and (2) and using this we consider an accelerated inexact Newton method for finding the unique PDS of Eq. (1). In Section 3 we will give numerical examples to show the behavior of the considered algorithms.

Throughout this paper $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix, $\rho(\cdot)$ is the spectral radius, $\|\cdot\|$ is the spectral norm, and $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ – infinity norm. We will say that X_S and X_L are a smallest and a largest PDSs of Eq. (2), respectively, when for all PDS X , we have $X_S \leq X \leq X_L$. Also, we will say that X_- and X_+ are a smallest and a largest Hermitian solutions of Eq. (1), respectively, when for all Hermitian solution X , we have $X_- \leq X \leq X_+$.

2 Iterative methods

In this section we present some iterative methods for finding the unique PDS of Eq. (1).

In the first consideration of equation (1) [3], the following *Fixed-point iteration (FPI)*

has been investigated:

$$\begin{cases} X_0 = Q, \\ X_{k+1} = Q + A^* X_k^{-1} A, \quad k = 0, 1, \dots \end{cases} \quad (3)$$

For this method we have

$$\begin{aligned} X_{2k-2} \leq X_{2k} \leq X_{2s+1} \leq X_{2s-1} \text{ for all } k, s \geq 1 \text{ and } \lim_{k \rightarrow \infty} X_k = X_+ [3], \\ \|X_{k+1} - X_+\| \leq \|X_+^{-1} A\|^2 \|X_k - X_+\|, \text{ for all } k \geq 0, \\ \text{and } \lim_{k \rightarrow \infty} \sup \sqrt[k]{\|X_k - X_+\|} \leq (\rho(X_+^{-1} A))^2 < 1 [8]. \end{aligned}$$

In addition to the above results for the convergence rate of FPI, Guo and Lancaster [8] propose *Newton's method* for Eq. (1):

$$\begin{cases} X_0 \text{ is close enough to } X_+, \\ X_{k+1} + (X_k^{-1} A)^* X_{k+1} X_k^{-1} A = Q + 2A^* X_k^{-1} A, \quad k = 0, 1, \dots \end{cases} \quad (4)$$

The Newton's method guarantees quadratic convergence if the initial guess X_0 must be chosen close to the largest solution X_+ . Hence, it can be used as an efficient correction method [8].

Meini [9] has studied a *Cyclic reduction method (CR)* for finding the solution X_+ of Eq. (1), which has quadratic rate of convergence:

$$\begin{cases} A_1 = A Q^{-1} A, \\ X_1 = Q + A^* Q^{-1} A, \\ Q_1 = Q + A Q^{-1} A^* + A^* Q^{-1} A, \\ A_{k+1} = A_k Q_k^{-1} A_k, \\ X_{k+1} = X_k - A_k^* Q_k^{-1} A_k, \\ Q_{k+1} = Q_k - A_k Q_k^{-1} A_k^* - A_k^* Q_k^{-1} A_k, \quad k = 1, 2, \dots \end{cases} \quad (5)$$

In Newton's method, at each iteration we have to solve a linear matrix equation of the following form

$$Y + C^* Y C = D. \quad (6)$$

A sufficient condition for the existence of a unique solution $\bar{Y}$ of Eq. (6) is $\tilde{D} - C^* \tilde{D} C > 0$ for a $\tilde{D} > 0$ (see [10, Proposition 3.1]). In this case it follows $\rho(C) < 1$ and the unique solution is given by

$$\bar{Y} = \sum_{j=0}^{\infty} (-1)^j (C^*)^j D C^j. \quad (7)$$

In [1] we consider an inexact Newton method for Eq. (1). Here we rewrite Eq. (4) the following way:

$$(X_{k+1} - X_k) + L_k^* (X_{k+1} - X_k) L_k = Q + L_k^* A - X_k, \quad (8)$$

where $L_k = X_k^{-1} A$ and solve it by using 2-Smith iteration [11]. This technique for construction an inexact Newton method can be find in [12] realised for non-symmetric Riccati equation.

Algorithm 1 (Inexact Newton method (INM)) [1, Algorithm 10]. Take X_0 sufficiently close to X_+ .

For $k = 0, 1, \dots$, compute
 $L_k = X_k^{-1}A$, $D_k = Q + L_k^*A - X_k$,
 take $C_{k,0} = L_k$, $Y_{k,1} = D_k - C_{k,0}^*D_kC_{k,0}$
 for $p = 1, 2, \dots, k$, compute
 $C_{k,p} = C_{k,p-1}^2$, $Y_{k,p+1} = Y_{k,p} + C_{k,p}^*Y_{k,p}C_{k,p}$,
 $X_{k+1} = X_k + Y_{k,k+1}$,
 until $\|Y_{k,k+1}\|_\infty \leq \text{tol}$, for a fixed error bound $\text{tol} > 0$.

Then $X_+ \approx X_{k+1}$.

The main problem with this method and the classical Newton's method is the choice of X_0 . In addition to the above method in [1], we propose an algorithm for Eq. (1) by using its relation to Eq. (2) and realisation of inexact Newton method for obtained equation.

Let X_+ be a unique PDS of Eq. (1). Then, by Woodbury formula, we have

$$\begin{aligned} X_+ &= Q + A^*X_+^{-1}A = Q + A^*(Q + A^*X_+^{-1}A)^{-1}A \\ &= Q + A^*[Q^{-1} - Q^{-1}A^*(X_+ + AQ^{-1}A^*)^{-1}AQ^{-1}]A \\ &= Q + A^*Q^{-1}A - A^*Q^{-1}A^*(X_+ + AQ^{-1}A^*)^{-1}AQ^{-1}A \end{aligned}$$

and

$$Y_L + B^*Y_L^{-1}B = P,$$

where $B = AQ^{-1}A$, $P = Q + A^*Q^{-1}A + AQ^{-1}A^*$, and $Y_L = X_+ + AQ^{-1}A^*$ is the largest solution of Eq. (2).

We note that the matrices B and P will be used further with the above meanings.

By using this relationship and investigated in [1] an inexact Newton method for Eq. (2) has been introduced following method:

Algorithm 2 (Accelerated Inexact Newton method (AINM)) [1, Algorithm 11]. Take $Y_0 = P$.

For $k = 0, 1, \dots$, compute
 $M_k = Y_k^{-1}B$, $D_k = P - M_k^*B - Y_k$,
 take $C_{k,0} = M_k$, $Z_{k,1} = D_k + C_{k,0}^*D_kC_{k,0}$,
 for $p = 1, 2, \dots, k$, compute
 $C_{k,p} = C_{k,p-1}^2$, $Z_{k,p+1} = Z_{k,p} + C_{k,p}^*Z_{k,p}C_{k,p}$,
 $Y_{k+1} = Y_k + Z_{k,k+1}$,
 until $\|Y_{k,k+1}\|_\infty \leq \text{tol}$, for a fixed error bound $\text{tol} > 0$.

Then $X_+ \approx Y_{k+1} - AQ^{-1}A^*$.

Fang et al. [2] by using same technique has obtained a relationship between equation $X - A^* \bar{X}^{-1} A = I$ ($X - A^* X^{-1} A = I$) and Eq. (2) and has investigated the considered equations. They have considered different well-known iterative methods for Eq. (2), which have used for finding the largest solution of the equation $X - A^* X^{-1} A = I$ (Eq. (1) in case of $Q = I$).

We now present some of the methods considered in [2] after slightly generalization for equation (1).

It follows the *Accelerated fixed-point iteration (AFPI)*:

$$\begin{cases} Y_0 = P, \\ Y_{k+1} = P - B^* Y_k^{-1} B, \quad k = 0, 1, \dots \end{cases} \quad (9)$$

Then, we have $Y_0 \geq Y_1 \geq Y_2 \geq \dots$ and $\lim_{k \rightarrow \infty} Y_k = X_+ + A Q^{-1} A^*$.

The next method which we consider is an *Inversion-free variant of AFPI (IFVAFPI)*:

$$\begin{cases} Y_0 = P, \quad 0 < Z_0 \leq P^{-1}, \\ Z_{k+1} = Z_k(2I - Y_k Z_k), \\ Y_{k+1} = P - B^* Z_{k+1} B, \quad k = 0, 1, \dots \end{cases} \quad (10)$$

Then, we have $Y_0 \geq Y_1 \geq Y_2 \geq \dots$, $Z_0 \leq Z_1 \leq Z_2 \leq \dots$, and $\lim_{k \rightarrow \infty} Y_k = X_+ + A Q^{-1} A^*$.

The last method we consider we call *Accelerated Newton's Method (ANM)*:

$$\begin{cases} Y_0 = P, \\ Y_{k+1} - (Y_k^{-1} A)^* Y_{k+1} Y_k^{-1} A = P - 2A^* Y_k^{-1} A, \quad k = 0, 1, \dots \end{cases} \quad (11)$$

Then, we have $Y_0 \geq Y_1 \geq Y_2 \geq \dots$ and $\lim_{k \rightarrow \infty} Y_k = X_+ + A Q^{-1} A^*$.

Remark 1 In above iterative methods the expressions $Z^{-1}A$ and $A^*Z^{-1}A$, where Z is a positive definite matrix, we compute following way: firstly, by Cholesky factorization $Z = R^*R$, we find upper triangular matrix R , after then solve the equations $R^*Y = A$ and $RL = Y$. Then we have $L = Z^{-1}A$ and $Y^*Y = A^*Z^{-1}A$.

3 Numerical examples

In this section, we conduct numerical experiments to compute unique PDS of Eq. 1 and compare the considered algorithms: Fixed-point iteration – FPI (3); Newton's method – NM (4); Cyclic reduction method – CR (5); Inexact Newton method – INM (Algorithm 1); Accelerated inexact Newton method – AINM (Algorithm 2); Accelerated fixed-point iteration – AFPI (10); Inversion-free variant of AFPI – IFVAFPI (10); and Accelerated Newton method – ANM (11). We report the number of required iterations (k) for converging. In our implementations, we stop all considered methods when the infinity norm of two successive iterates is less than given tolerance ($\|X_k - X_{k-1}\|_\infty \leq tol$). Let us $res(X) = \|Q - A^*X^{-1}A - X\|_\infty$.

Example 1 [8, Example 7.4] We consider Eq. (1) with matrix coefficients

$$A = \begin{pmatrix} 50 & 20 \\ 10 & 60 \end{pmatrix}, \quad Q = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}.$$

For the matrices in Example 1, we have $\rho(Q^{-1}A) \approx 27$. The convergence of NM (4) with $X_0 = Q$ is not guaranteed and INM (Algorithm 1) is inapplicable. The results of the experiment are presented in Table 1.

Table 1: Numerical results of Example 1 with $tol = 10^{-8}$

Method	k	$\ X_k - X_{k-1}\ _\infty$	$res_-(X_k)$
FPI-(3))	422	$9.7486e - 09$	$9.1448e - 09$
NM (4)	10	$1.7764e - 14$	$2.4869e - 14$
CR (5)	9	$2.4412e - 11$	$1.9398e - 12$
AINM (Algorithm 2)	8	$2.0206e - 09$	$4.8864e - 10$
AFPI (10)	187	$8.3695e - 09$	$1.1107e - 07$
IFVAFPI (10)	188	$8.3996e - 09$	$1.0995e - 07$
ANM (11)	8	$9.0296e - 10$	$4.6469e - 12$

The most efficient algorithm is CR (5). Using the relationship between Eq. (1) and Eq. (2) and subsequently using methods for Eq. (2) gives a positive result. AFPI (10) and its inverse free variant - IFVAFPI (10) have more than half the number of iterations than FPI-(3)). AINM (Algorithm 2) and ANM (11) are always applicable with $X_0 = Q$, but when performing the calculations according to Remark 1. In this case, the symmetry of $A^T Z^{-1} A$ is preserved. For example, if we calculate $L = Z^{-1} A$ and $A^T Z^{-1} A = L^* A$, then the convergence of ANM fails because the symmetry is not preserved due to rounding errors.

The methods NM (4) and INM (Algorithm 1) are suitable to be used as an efficient correction method [8, 1]. After 75 iterations by FPI-(3)) for Example 1, we have $\rho(X_{73}^{-1} A) = 0.9448$ and $\rho(X_{74}^{-1} A) = 0.9979$. Then, using X_{75} as an initial value in NM (4) and INM (Algorithm 1), we obtain the results in Table 2.

Table 2: Numerical results of Example 1 after 75 iterations with FPI-(3)

Method	k	$\ X_{75+k} - X_{74+k}\ _\infty$	$res(X_{75+k})$
NM (4)	4	$1.8474e - 13$	$1.7764e - 14$
INM (Algorithm 1)	8	$1.6261e - 13$	$3.9080e - 14$

Example 2 [13, Example 5] We consider Eq. (1) with $Q = I$ and

$$A = \begin{pmatrix} -3.47 & 3.47 \\ -2.89 & -3.47 \end{pmatrix}.$$

For the matrix A in Example 2, we have $\rho(A) \approx 4.7$. The results of the experiment are presented in Table 3. The properties of the methods under consideration, which are manifested for Example 2, are identical to those in Example 1.

Table 3: Numerical results of Example 2 with $tol = 10^{-8}$

Method	k	$\ X_k - X_{k-1}\ _\infty$	$res_-(X_k)$
FPI-(3))	101	$8.2950e - 09$	$6.8640e - 09$
NM (4)	10	$1.7764e - 14$	$2.4869e - 14$
CR (5)	7	$1.4991e - 11$	$1.4766e - 14$
AINM (Algorithm 2)	6	$3.4974e - 11$	$1.7764e - 15$
AFPI (10)	47	$7.7036e - 09$	$2.3959e - 08$
IFVAFPI (10)	48	$6.5516e - 09$	$2.2264e - 08$
ANM (11)	6	$2.3184e - 11$	$[8.4377e - 15]$

After 15 iterations by FPI-(3)) for Example 2, we have $\rho(X_{13}^{-1}A) = 0.8199$ and $\rho(X_{14}^{-1}A) = 0.9676$. Then, using X_{15} as an initial value in NM (4) and INM (Algorithm 1), we obtain the results in Table 4.

Table 4: Numerical results of Example 2 after 15 iterations with FPI-(3)

Method	k	$\ X_{15+k} - X_{14+k}\ _\infty$	$res(X_{15+k})$
NM (4)	4	$4.1225e - 12$	$3.5527e - 15$
INM (Algorithm 1)	6	$4.3683e - 13$	$9.9920e - 16$

Example 3 [2, Example 5.2] We consider Eq. (1) with $Q = I$ and

$$A = 2 \text{rand}(4) - 1 + i(2 \text{rand}(4) - 1),$$

where rand is a function in Matlab and $i = \sqrt{-1}$.

For the matrix A in Example 3, we have $\rho(A) \approx 1.9$. The results of the experiment are presented in Table 5. The convergence of NM (4) with $X_0 = Q$ is not guaranteed. The results are identical to the above.

Table 5: Numerical results of Example 2 with $tol = 10^{-8}$

Method	k	$\ X_k - X_{k-1}\ _\infty$	$res_-(X_k)$
FPI-(3))	34	$6.2204e - 09$	$3.3123e - 09$
NM (4)	11	$3.0430e - 14$	$1.5345e - 14$
CR (5)	6	$2.9098e - 17$	$1.8016e - 15$
AINM (Algorithm 2)	5	$5.6156e - 14$	$1.8029e - 15$
AFPI (10)	17	$2.9083e - 09$	$1.7733e - 09$
IFVAFPI (10)	18	$4.1877e - 09$	$2.4888e - 09$
ANM (11)	5	$6.3674e - 14$	$2.8966e - 14$

After 3 iterations by FPI-(3)) for Example 3, we have $\rho(X_1^{-1}A) = 0.4234$ and $\rho(X_2^{-1}A) = 0.9798$. Then, using X_3 as an initial value in NM (4) and INM (Algorithm 1), we obtain the results in Table 6.

Table 6: Numerical results of Example 2 after 3 iterations with FPI-(3)

Method	k	$\ X_{3+k} - X_{2+k}\ _\infty$	$res(X_{3+k})$
NM (4)	5	$1.4781e - 14$	$9.9715e - 15$
INM (Algorithm 1)	5	$1.2369e - 15$	$1.2192e - 15$

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