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An effective approach to solve a nonsymmetric algebraic Riccati equation

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Abstract. The MALI iteration (has developed by Guan) is an effective iteration to compute a nonnegative solution. We propose a decoupled modification of Guan's procedure to compute the minimal nonnegative solution of the M-matrix algebraic Riccati equation. The convergence properties of the proposed iteration are derived and a sufficient condition for convergence is derived. The performance of the proposed algorithm is illustrated on several numerical examples.

Key Words: numerical iterative methods, M-matrix, minimal nonnegative solution.

1 Introduction

The equation

$$\mathcal{R}(X) := \ XCX - XD - AX + B = 0\,. \tag{1}$$

is investigated in [5], where D, B, C and A are real matrices of dimensions $m\times m, m\times n, n\times m$ and $n \times n$, respectively. The block matrix $K = \begin{pmatrix} A & -C \\ -B & D \end{pmatrix}$ is an M-matrix.

The general nonsymmetric matrix Riccati equation associated with M-matrices has many applications - in the Markov chains [6], in the transport theory [4] and many others. Nonsymmetric Riccati equation (1) arises from the game theory and more specially from the investigation of the open-loop Nash linear quadratic differential game [7, 1, 8]. A more general problem on connected to the properties of the stabilising solution of the game theoretic algebraic Riccati equation is investigated in [2, 3]. The solution of practical interest is the minimal nonnegative solution of (1).

There are many numerical methods up to now proposed for the minimal nonnegative solution of (1) with a nonsingular M-matrix. An effective method called alternately linearized implicit iteration method (ALI) was proposed and investigated in [11, 9, 10]. A new alternately linearized implicit iteration method (NALI) for computing the minimal nonnegative solution of (1) is introduced in [10].

The notation $\mathbf{R}^{s\times q}$ stands for $s \times q$ real matrices. In this investigation we exploit the properties of nonnegative matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{m\times n}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We use an elementwise order relation. The inequality $P \geq Q(P > Q)$ for $P = (p_{ij}), Q = (q_{ij})$ means that $p_{ij} \geq q_{ij}(p_{ij} > q_{ij})$ for all indexes i and j. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$. The notation I means an unit $n \times n$ matrix.

2 Preliminary statements

There two iterative methods based on the alternately linearized implicit iteration method - ALI iterative method and A modified ALI iteration method (named NALI method). Guan [5] has described the following iterative methods.

I. ALI iterative method: $X^{(0)} = 0 \in R^{n\times n}, \mu > 0$ is given parameter:

$$\begin{aligned} &Y^{(k)}(\mu I_n + D - CX^{(k)}) = (\mu I_n - A)X^{(k)} + B \\ &(\mu I + A - Y^{(k)}C)X^{(k+1)} = Y^{(k)}((\mu I_n - D) + B \end{aligned} \tag{2}$$

II. NALI iterative method: $X^{(0)} = 0 \in R^{n\times n}, \mu > 0$ is given parameter:

$$\begin{aligned} &Y_k(\mu I + D) = (\mu I - A + X_k C)X_k + B \\ &(\mu I + A)X_{k+1} = Y_k((\mu I - D + CY_k) + B \end{aligned} \tag{3}$$

Guan [5] has proposed the MALI iterative method: $A = (a_{ij}), D = (d_{ij}), X^{(0)} = 0 \in R^{n\times n}$. The matrix A is transformed $A = L_A - U_A$, where L_A is the lower triangular part of A and U_A is the strictly upper triangular part of A. The matrix D is received the type $D = L_D - U_D$ in the same way.

III. The iteration scheme is:

$$\begin{aligned} &Y^{(k)}(\alpha I + L_D) = (\alpha I - A + X^{(k)}C)X^{(k)} + X^{(k)}U_D + B\,, \quad \alpha \geq max_i\ (a_{ii}) \\ &(\delta I + L_A)X^{(k+1)} = Y^{(k)}(\delta I - D + CY^{(k)}) + U_A Y^{(k)} + B\,, \quad \delta \geq max_i\ (d_{ii})\,. \end{aligned} \tag{4}$$

The convergence analysis of the MALI iteration method and numerical experiments are executed by Guan.

Here, we slightly modify iteration 4.

New modifications of the MALI iterative method. $A = (a_{ij}), D = (d_{ij}), X^{(0)} = 0 \in R^{n\times n}$. We transform $D = L_D - U_D$, where L_D is the lower triangular part of D and U_D is the strictly upper triangular part of D. Compute

$$\gamma = max\{\alpha, \beta\}\,, \quad \alpha \geq max_i\ (a_{ii}) \ \ \beta \geq max_i\ (d_{ii})\,.$$

The iteration has the form

$$\begin{aligned} &Y^{(k)}(\gamma I + L_D) = (\gamma I - A + X^{(k)}C)X^{(k)} + X^{(k)}U_D + B\,, \\ &(\gamma I + A)X^{(k+1)} = Y^{(k)}(\gamma I - D + CY^{(k)}) + B\,, \quad . \end{aligned} \tag{5}$$

Lemma 1 *We construct the matrix sequences $\{X^{(k)}, Y^{(k)}\}_{k=0}^{\infty}$ using ((5)) with initial values $X^{(0)} = 0$. Then for any positive k , the following equalities hold:*

$$\text{(i)} \quad \mathcal{R}(X^{(k)}) \quad = (Y^{(k)} - X^{(k)})(\gamma I + L_D)\,,$$

$$\begin{aligned}\text{(ii)} \quad \mathcal{R}(Y^{(k)}) = \; & (\gamma I - A + X^{(k)}C)(Y^{(k)} - X^{(k)}) \\ & + (Y^{(k)} - X^{(k)})(U_D + CY^{(k)})\,,\end{aligned}$$

$$\text{(iii)} \; \mathcal{R}(Y^{(k)}) = (\gamma I + A)(X^{(k+1)} - Y^{(k)})\,,$$

$$\begin{aligned}\text{(iv)} \; \mathcal{R}(X^{(k+1)}) = \; & (X^{(k+1)} - Y^{(k)})(\gamma I - D + CY^{(k)}) \\ & + X^{(k+1)}C\,(X^{(k+1)} - Y^{(k)})\,,\end{aligned}$$

$$\begin{aligned}\text{(v)} \; \mathcal{R}_i(\hat{X}) \; = \; & (Y^{(k)} - \hat{X})(\gamma I + L_D) + (\gamma I - A + \hat{X}C)(\hat{X} - X^{(k)}) \\ & + (\hat{X} - X^{(k)})(U_D + CX^{(k)})\,.\end{aligned}$$

$$\begin{aligned}\text{(vi)} \; \mathcal{R}(\hat{X}) = \; & (\gamma I + A)(X^{(k+1)} - \hat{X}) \\ & + (\hat{X} - Y^{(k)})(\gamma I - D + CY^{(k)}) + \hat{X}C\,(\hat{X} - Y^{(k)})\,.\end{aligned}$$

Proof. The proof is completed by a direct calculation.
The main theorem is

Theorem 2 *Assume the matrix A is an M-matrix and $B \geq 0, C \geq 0$, and there exists $\mu > 0$, such that $(\mu I + A)$ is an M-matrix and $\mu I - A$ is nonpositive.*
Assume there exist a nonnegative matrix $\hat{X}$, such that $\mathcal{R}(\hat{X}) \leq 0$ . .
The matrix sequence $\{X^{(k)}\}_{k=0}^{\infty}$ defined by ((5)) satisfy the following properties:
(i) $\hat{X} \geq X^{(k+1)} \geq Y^{(k)} \geq X^{(k)}$ *for* $k = 0, 1, \ldots$*;*

(ii) $\mathcal{R}(X(k)) \geq 0\,, \quad \mathcal{R}(Y^{(k)}) \geq 0\,, \quad \mathcal{R}(X^{(k+1)}) \geq 0\,, \quad k = 0, 1, \ldots\,.$

(iii) *The matrix sequence $\{X^{(k)}\}_{k=0}^{\infty}$ converges to a nonnegative solution $\tilde{X}$ to the Riccati equation $\mathcal{R}(X) = 0$.*

Proof. We apply the decomposition of the matrix coefficient $D = L_D - U_D$, where L_D is the lower triangular part of D and U_D is the strictly upper triangular part of D. We remark $U_D \geq 0$. We begin with the facts that $(\gamma I + A)^{-1} \geq 0$, and $(\gamma I + L_D)^{-1} \geq 0$. We construct the matrix sequences $\{X^{(k)}, Y^{(k)}\}_{k=0}^{\infty}$ applying recursive equations ((5)) with $X^{(0)} = 0$ and $\gamma_i > 0$.

For $k = 0$ we obtain $Y^{(0)}(\gamma I + L_D) = B \geq 0$ and thus $Y^{(0)} = B(\gamma I + L_D)^{-1} \geq 0$. And $Y^{(0)} \geq X^{(0)} = 0$. In addition, $\mathcal{R}(X^{(0)}) = B \geq 0$.

Applying Lemma 1 (ii), we get $(\gamma I - A \geq 0)$

$$\mathcal{R}(Y^{(0)}) = (\gamma I - A)Y^{(0)} + Y_i^{(0)}(U_D + CY^{(0)}) \geq 0\,.$$

We compute $X^{(1)}$ applying the recursive equation (5). We have

$$(\gamma I + A)X^{(1)} = W^{(0)} \geq 0\,,$$

where

$$W^{(0)} := Y^{(0)}(\gamma I - D + CY^{(0)}) + B\,.$$

Since $(\gamma I + A)^{-1} \geq 0$, we obtain $X^{(1)}$ is nonnegative.

Applying Lemma 1 (iii), we get

$$(X^{(1)} - Y^{(0)}) = (\gamma I + A)^{-1}\,\mathcal{R}(Y^{(0)}) \geq 0\,.$$

According to Lemma 1 (iv) we induce

$$\begin{aligned}\mathcal{R}(X^{(1)}) = (X^{(1)} - Y^{(0)})(\gamma I - D + CY^{(0)})\\ +X^{(1)}C(X^{(1)} - Y^{(0)}) \;\geq 0\,,\end{aligned}$$

because $\gamma\, I - D \geq 0, X^{(1)} \geq Y^{(0)} \geq X^{(0)}$.

In order to prove $\hat{X} \geq X^{(1)}$ we consider equality Lemma 1 (v)

$$\begin{aligned}\mathcal{R}(\hat{X}) \;&= (Y^{(0)} - \hat{X})(\gamma I + L_D)\\ &+(\gamma I - A + \hat{X}C)\hat{X} + \hat{X}U_D \geq 0\,,\end{aligned}$$

Note that $\gamma I - A \geq 0\,,\; U_D \geq 0\,,\;$.

$$(Y^{(0)} - \hat{X}) = H^{(0)}\,(\gamma I + L_D)^{-1} \leq 0\,,$$

because

$$H^{(0)} := \mathcal{R}(\hat{X}) - (\gamma I - A + \hat{X}C)\hat{X} - \hat{X}U_D \;\leq 0\,.$$

Thus $\hat{X} \geq Y^{(0)}$. Moreover, applying equality Lemma 1 (vi) with $k = 0$ we obtain

$$\begin{aligned}(\gamma I + A)(X^{(1)} - \hat{X}) = \mathcal{R}(\hat{X}) - (\hat{X} - Y^{(0)})(\gamma I - D + CY^{(0)})\\ -\hat{X}C(\hat{X} - Y^{(0)})\,.\end{aligned}$$

We infer $\hat{X} \geq X^{(1)}$.

So, we have proved inequalities (i) - (ii) for $k = 0$.

We assume that the inequalities (i) - (ii) hold for $k = 0, 1, \ldots, r$. We know matrices $X^{(r+1)}$ with the properties:

$$\hat{X} \geq X^{(r+1)} \geq Y^{(r)} \geq X^{(r)}\,,$$

and

$$\mathcal{R}(X^{(r)}) \geq 0\,, \quad \mathcal{R}(Y^{(r)}) \geq 0\,, \quad \mathcal{R}(X^{(r+1)}) \geq 0\,.$$

We will prove the inequalities (i) - (ii) for $k = r + 1$.

We compute $Y^{(r+1)}$ via ((5)), i.e.

$$Y^{(r+1)} = [(\gamma I - A + X^{(r+1)}C)X^{(r+1)} + X^{(r+1)}U_D + B]\,(\gamma I + L_D)^{-1} \geq 0\,.$$

According to Lemma 1 (i) we extract

$$Y^{(r+1)} - X^{(r+1)} = \mathcal{R}(X^{(r+1)})(\gamma I + L_D)^{-1} \geq 0\,,$$

From Lemma 1 (ii), we conclude

$$\begin{aligned}\mathcal{R}(Y^{(r+1)}) = \ & (\gamma I - A + X^{(r+1)}C)(Y^{(r+1)} - X^{(r+1)}) \\ & +(Y^{(r+1)} - X^{(r+1)})(U_D + CY^{(r+1)}) \ \geq 0\,,\end{aligned}$$

We compute $X^{(r+2)}$ via the second equation of (5). Consider the equality (iii) of Lemma 1 for $k = r+1$. We write down:

$$X^{(r+2)} - Y^{(r+1)} = (\gamma I + A)^{-1}\,\mathcal{R}(Y^{(r+1)}) \geq 0\,.$$

Next, we apply of Lemma 1 (iv) for

$$\begin{aligned}\mathcal{R}(X^{(r+2)}) = & (X^{(r+2)} - Y^{(r+1)})(\gamma I - D + CY^{(r+1)}) \\ & +X^{(r+2)}C(X^{(r+2)} - Y^{(r+1)}) \geq 0\,.\end{aligned}$$

Thus $\mathcal{R}(X^{(r+2)}) \geq 0\,.$

In order to prove $\hat{X} \geq X^{(r+2)}$ we consider equality Lemma 1 (v)

$$\begin{aligned}\mathcal{R}(\hat{X}) \ \ & = (Y^{(r+1)} - \hat{X})(\gamma I + L_D) \\ & +(\gamma I - A + \hat{X}C)(\hat{X} - X^{(r+1)}) - (\hat{X} - X^{(r+1)})(U_D + CX^{(r+1)})\,,\end{aligned}$$

Note that $\gamma I - A \geq 0\,,\ U_D \geq 0\,,\ .$ Then

$$Y^{(r+1)} - \hat{X} = H^{(r+1)}\,(\gamma I + L_D)^{-1} \leq 0\,,$$

because $H^{(r+1)} \leq 0$, and

$$\begin{aligned}H^{(r+1)} := & \mathcal{R}(\hat{X}) - (\gamma I - A + \hat{X}C)(\hat{X} - X^{(r+1)}) \\ & -(\hat{X} - X^{(r+1)})(U_D + CX^{(r+1)})\,,\end{aligned}$$

Thus $\hat{X} \geq Y^{(r+1)}\,.$

Further on, taking into account Lemma 1 (vi) we obtain

$$X_i^{(r+2)} - \hat{X} = (\gamma I + A)^{-1}\,T^{(r+1)} \leq 0\,,$$

because $T^{(r+1)} \leq 0$, and

$$\begin{aligned}T^{(r+1)} := & \mathcal{R}(\hat{X}) - (\hat{X} - Y^{(r+1)})(\gamma I - D + CY^{(r+1)}) \\ & -\hat{X}C(\hat{X} - Y^{(r+1)})\,.\end{aligned}$$

We infer $\hat{X} \geq X^{(r+2)}\,.$

Hence, the induction process has been completed. Thus the matrix sequence $\{X^{(k)}\}_{k=0}^{\infty}$ are nonnegative, monotonically increasing and bounded from above by ($\hat{X}$ (in the element-wise ordering). We denote $\lim_{k\to\infty}(X^{(k)}) = (\tilde{X})$. By taking the limits in (5) it follows that $(\tilde{X})$ is a solution of $\mathcal{R}(X) = 0$ with the property $\tilde{X}$.

The theorem is proved.

3 Numerical Experiments

Following [12], we apply the relative residual error (RES) defined by

$$RES = \frac{\mathcal{R}(X_k)}{\mathcal{R}(X_0)}\,.$$

In our computations $\|\mathcal{R}(X_0)\| = \|B\|$ because $X_0 = 0$ and use the stop criterion $RES \leq tol$. Algorithms stop for different values of tol (see Table 1).

We apply the iteration methods (4) and (5) for computing the minimal nonnegative solution of equation (1). We take the values of n as follows: $n = 12, 24, 32$.

Example 1. We introduce an example with the matrix coefficients with different values of n: $A = (a_{ij}), D = (d_{ij})$, and $B = 0.75\,I_n,\ C = 0.92\,I_n$.

$$A = \begin{cases} a_{ii} = 4.\,, & i = 1, \ldots, n\,, \\ a_{i,i+1} = -1\,, & i = 1, \ldots, n-1\,, \\ a_{i+1,i} = -0.1\,, & i = 1, \ldots, n-1\,, \\ a_{i,i+2} = -0.55\,, & i = 1, \ldots, n-2\,, \\ a_{i+2,i} = -0.525\,, & i = 1, \ldots, n-2\,, \end{cases}$$

$$D = \begin{cases} d_{ii} = 2.\,, & i = 1, \ldots, n\,, \\ d_{i,j} = a_{ij}/5\,, & i \neq j, i, j = 1, \ldots, n\,, \end{cases}$$

Example 2. We introduce an example with the matrix coefficients with different values of n: $A = (a_{ij}), D = (d_{ij})$, and $B = 0.75\,I_n,\ C = 0.92\,I_n$.

$$A = \begin{cases} a_{ii} = 4.\,, & i = 1, \ldots, n\,, \\ a_{i,i+1} = -1\,, & i = 1, \ldots, n-1\,, \\ a_{i+1,i} = -0.33\,, & i = 1, \ldots, n-1\,, \\ a_{i,i+2} = -0.55\,, & i = 1, \ldots, n-2\,, \\ a_{i+2,i} = -1.925\,, & i = 1, \ldots, n-2\,, \\ a_{1n} = -0.15\,, a_{n1} = -1.7\,, & \end{cases}$$

$$D = \begin{cases} d_{ii} = 2.\,, & i = 1, \ldots, n\,, \\ d_{i,j} = a_{ij}/5\,, & i \neq j, i, j = 1, \ldots, n\,, \end{cases}$$

Example 3. We introduce an example with the matrix coefficients with different values of n: $A = (a_{ij}), D = (d_{ij})$, and $B = 0.75\,I_n,\ C = 0.92\,I_n$.

$$A = \begin{cases} a_{ii} = 4.\,, & i = 1, \ldots, n\,, \\ a_{i,i+1} = -1\,, & i = 1, \ldots, n-1\,, \\ a_{i+1,i} = -0.33\,, & i = 1, \ldots, n-1\,, \\ a_{i,i+2} = -0.55\,, & i = 1, \ldots, n-2\,, \\ a_{i+2,i} = -1.925\,, & i = 1, \ldots, n-2\,, \\ a_{1n} = -0.005\,, a_{n1} = -1\,, & \end{cases}$$

$$D = \begin{cases} d_{ii} = 2.\,, & i = 1, \ldots, n\,, \\ d_{i,j} = a_{ij}/5\,, & i \neq j, i, j = 1, \ldots, n\,, \end{cases}$$

Table 1. Experiments with (4) and (5) for 100 runs

	(4)		(5)	
	Example 1 , $tol = 1.0e - 14$			
n	It	CPU	It	CPU
18	25	0.06s	22	0.05s
32	26	0.15s	23	0.14s
48	27	0.44s	23	0.43 s
	Example 2 , $tol = 1.0e - 14$			
n	It	CPU	It	CPU
18	128	0.29s	105	0.25s
32	328	1.7s	272	1.48s
36 $tol = 1.0e - 12$	720	5.6s	600	4.8s
48	–	— s	—	s
	Example 3 , $tol = 1.0e - 14$			
n	It	CPU	It	CPU
18	119	0.28s	98	0.22s
32	202	1.0s	166	0.91s
48	330	5.8s	272	4.6s
56	561	14.1s	467	11.5s

4 Conclusion

The main conclusion is that the new iterative formula constructs an effective algorithm for computing the nonnegative solution of the matrix equation $\mathcal{R}(X) = 0$. Moreover, there are examples where iteration (5) is faster than iteration (4) (See Table 1).

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