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Improved Iterative Methods for Computing the Nash Equilibrium in Positive Systems

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Abstract. In this paper, we consider the nonsymmetric algebraic Riccati equation arising in computing a Nash Equilibrium two-player linear quadratic differential games for positive linear systems with an open loop information structure. We improve the existing methods for finding the Nash equilibrium, such as the Newton method and Sylvester method. The proposed improvements follow the new ideas. Moreover, the convergence properties of the proposed modifications are investigated and the sufficient conditions to apply the modifications are derived. The performances of the proposed algorithms are illustrated on some numerical examples.

Key Words: nonsymmetric algebraic Riccati equations, numerical iteration methods, open loop Nash equilibrium, positive systems.

1 Introduction

We consider the Riccati equation

$$0 = -\begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} A - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad (1)$$

where $-A$ is a $n \times n$ Z-matrix, B_j is an $n \times m_j$ nonnegative matrix, $S_j = B_j\, R_{jj}^{-1}\, B_j^T$ ($S_j = S_j^T$) is a nonpositive matrix, Q_j is an $n \times n$ symmetric nonnegative matrix, R_{jj} is an $m_j \times m_j$ negative definite matrix for $j = 1, 2$ and X_1, X_2 are $n \times n$ unknown matrices. In this paper we use the following notations: $\mathbf{R^{n\times s}}$ stands for $n \times s$ real matrices. The inequality $X \geq 0$ ($X > 0$) means that all elements of the matrix X are real nonnegative (positive) and we call the matrix X nonnegative (positive). For the matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) hold for all indexes i and j. An $n \times n$ matrix A is called a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be presented as $A = sI_n - N$ with N being nonnegative matrix, and it is called a nonsingular M-matrix if $s > \rho(N)$, where $\rho(N)$ is the spectral radius of N. A matrix A is called asymptotically stable if the eigenvalues of A have a negative real part. A symmetric matrix A is called positive definite (semidefinite) matrix if all eigenvalues are positive (nonnegative).

We exploit the fact that the following statements are equivalent:
(a) A is a nonsingular M-matrix;
(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;
(c) A is asymptotically stable ,
for a Z-matrix $(-A)$.
We apply a property of the matrix equation $AXB=C$, i.e. it is equivalent to the linear system $(B^T \otimes A)$ vec X= vec C , where the sign $\otimes$ denotes the Kronecker matrix product and the vec operator arranges the columns of a matrix into a column vector.

1.1 The Newton method (NM)

The Newton method for finding the Nash equilibrium of a nonsymmetric algebraic matrix Riccati equation in a two-player linear-quadratic differential game is proposed in [4, 1]. The Newton method (NM) is introduced by G. Jank and D. Kremer in [4] with following iteration

$$(D - K_i S)K_{i+1} + K_{i+1}(A - SK_i) = -Q - K_i S K_i,\ i = 0, 1, 2... \qquad (2)$$

where

$$D = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix},\ Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},\ S = (S_1 \quad S_2)\ and\, K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

The following theorem proves that the Newton matrix sequence converges monotonically to a nonnegative solution.

The right side of the equation (1) we denote with $\mathcal{R}(X)$.

Theorem 1 (Theorem 5, [4]) *Suppose additionally for the nonsymmetric Riccati equation positive system*

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2,\ x(0) = x_0, \qquad (3)$$

that the matrix $-A$ is an M-matrix and $Q_i \geq 0, i = 1, 2$ and $S \leq 0$. Assume further that there exists a $P \geq 0$, such that $\mathcal{R}(P) > 0$, then the Newton sequence $\{X_i\}_{i \in N}$ initialized with $X_0 = 0$ is well defined and converges monotonically to a solution $X \geq 0$. X is the smallest solution in the set of all nonnegative solutions.

1.2 The Linearized Newton method (LNM)

The linearized Newton method is used by C. Ma and H. Lu in [3]. Set $Z_0 = 0$, we compute matrix sequences $\{Y_i\}$ and $\{Z_i\}$ via following iterations

$$Y_{i+1}(\gamma I_n + A - SZ_i) = (\gamma I_{2n} - D)Z_i - Q \qquad (4)$$

$$(\gamma I_{2n} + D - Y_{i+1}S)Z_{i+1} = Y_{i+1}(\gamma I_n - A) - Q, \qquad (5)$$

for $i = 0, 1, 2, ...$ and $\gamma < 0$, as sequence $\{Z_i\}$ converge to the solution $\tilde{K}$, when i converge to the infinity. [3]

2 Improved methods and convergence properties

2.1 The Newton Method with Parameters (NMP)

We use iteration (2) and written it in the form

$$-K^{(i+1)}(A - SK^{(i)}) - (D - K^{(i)}S)K^{(i+1)} = Q + K^{(i)}SK^{(i)},\ i = 0, 1, 2...,$$

and we produce improved iteration

$$K^{(i+1)}(\beta I_n + A - SK^{(i)}) + (\alpha I_{2n} + D - K^{(i)}S)K^{(i+1)} = -Q - K^{(i)}SK^{(i)} + (\alpha + \beta)K^{(i)}, \quad (6)$$

where α and β are negative numbers, $i = 0, 1, 2....$

We introduce a cost functional

$$J_i(u_1, u_2) = \int_0^\infty (x^T Q_i x + \sum_{j=1}^{2} u_j^T R_{ij} u_j)dt,$$

for $i = 1, 2$, which is minimized and input function u_i for $i = 1, 2$ is strategy for each player. We introduce a matrix function

$$P(X) = -DX - XA - Q + XSX.$$

Lemma 2 *Suppose that $-A \in R^{n\times n}$ is a nonsingular M-matrix, β is negative real number ($-\beta > 0$) and $C = -\beta I_n - A$, then C is a nonsingular M-matrix. (see Lemma 2.4 in [4]).*

Proof: Since $-A \in R^{n\times n}$ is a nonsingular M-matrix, then there exist $s > 0,\ N \geq 0$, such that $-A = sI_n - N$ and $s > \rho(N)$. We have

$$C = -\beta I_n - A = -\beta I_n + sI_n - N = (s - \beta)I_n - N,\ s - \beta > \rho(N).$$

Therefore C is a nonsingular M-matrix. The proof ends. □ For the convergence of the method we prove the following theorem:

Theorem 3 *For the positive system (3), suppose that the matrix $-A$ is an M-matrix. Assume that the matrix $Q \geq 0$ and $S \leq 0$ ($S_j = B_j R_{jj}^{-1} B_j^T, j = 1, 2$). Assume further that there exists a $\hat{\mathbf{X}} = \begin{pmatrix} \hat{K}_1 \\ \hat{K}_2 \end{pmatrix} \geq 0$, such that $P(\hat{\mathbf{X}}) > 0$, then the Newton sequence $\{K^{(i)}\}_{i\in N}$, $K^{(0)} = 0$, converges monotonically to a solution $K \geq 0$. The solution K is the smallest solution in the set of all nonnegative solutions.*

Proof:

$$P\left(\tilde{\mathbf{X}}\right) > 0 \ \Rightarrow \ -D\tilde{\mathbf{X}} - \tilde{\mathbf{X}}A - Q + \tilde{\mathbf{X}}S\tilde{\mathbf{X}} > 0 \ \Rightarrow$$

$$\tilde{\mathbf{X}}S\tilde{\mathbf{X}} > \tilde{\mathbf{X}}A + D\tilde{\mathbf{X}} + Q. \quad (7)$$

The initial matrix $K^{(0)} = 0$, the first element $K^{(1)}$ of the Newton sequence we calculate from the Sylvester equation

$$-K^{(i+1)}(\beta I_n + A) - (\alpha I_{2n} + D)K^{(i+1)} = Q,$$

which can be written as a linear system of equations in the form

$$[(-\beta I_n - A)^T \otimes I_{2n} + I_n \otimes (-\alpha I_{2n} - D)] \operatorname{vec} K^{(1)} = \operatorname{vec} Q,$$

where vec : $R^{i\times j} \to R^{ij\times 1}$. Let

$$L^{(i)} := [(-\beta I_n - A + SK^{(i)})^T \otimes I_{2n} + I_n \otimes (-\alpha I_{2n} - D + K^{(i)}S)],$$

$i = 0, 1, 2, ...,$ where α and β are negative numbers. For $i = 0$ we write the equation as

$$L^{(0)}\, vec\, K^{(1)} = vec\, Q.$$

As $-A$ is an M-matrix ($(-A)^{-1} \geq 0$), we apply Lemma 2 and hence $-\beta I_n - A$ is an M-matrix. The matrix

$$(-D)^{-1} = \begin{pmatrix} ((-A)^T)^{-1} & 0 \\ 0 & ((-A)^T)^{-1} \end{pmatrix} \geq 0,$$

hence $-D$ is an M-matrix. Since $\alpha < 0$ combine to Lemma 2, we obtain that $-\alpha I_{2n} - D$ is an M-matrix. Hence $L^{(0)}$ is an M-matrix, which is equivalent on $(L^{(0)})^{-1} \geq 0$, we conclude that $K^{(1)} \geq 0$.

We will prove by induction the following statements for $i = 0, ...$:
(A) $K^{(i)} \leq K^{(i+1)}$;
(B) $K^{(i)} \leq \hat{\mathbf{X}}$;
(C) $L^{(i)}$ is an M-matrix.
We will prove these properties for $i = 0$:
(A) $K^{(0)} = 0$ and we prove that $K^{(1)} \geq 0$, hence $0 = K^{(0)} \leq K^{(1)}$.
(B) $K^{(0)} = 0$ and we assume that $\hat{\mathbf{X}} \geq 0$, hence $0 = K^{(0)} \leq \hat{\mathbf{X}}$.
(C) above we have proven that $L^{(0)}$ is an M-matrix.
Assume that $K^{(i)} \leq K^{(i+1)}$, $K^{(i)} \leq \hat{\mathbf{X}}$ and $L^{(i)}$ is an M-matrix for $i = k > 0$. We will prove the statements (A)-(B)-(C) for $i = k + 1$.
To prove statement (B) for $i = k + 1$ we use (6) and $P(\hat{\mathbf{X}}) > 0$. We consider the matrix

$$\begin{aligned}
&(\hat{\mathbf{X}} - K^{(k+1)})(\beta I_n + A - SK^{(k)}) + (\alpha I_{2n} + D - K^{(k)}S)(\hat{\mathbf{X}} - K^{(k+1)}) \\
&= \beta\hat{\mathbf{X}} + \hat{\mathbf{X}}A - \hat{\mathbf{X}}SK^{(k)} + \alpha\hat{\mathbf{X}} + D\hat{\mathbf{X}} - K^{(k)}S\hat{\mathbf{X}} + Q + K^{(k)}SK^{(k)} - (\alpha + \beta)K^{(k)} \\
&< \hat{\mathbf{X}}S\hat{\mathbf{X}} + (\alpha + \beta)(\hat{\mathbf{X}} - K^{(k)}) - \hat{\mathbf{X}}SK^{(k)} - K^{(k)}S\hat{\mathbf{X}} + K^{(k)}SK^{(k)} \\
&= (\hat{\mathbf{X}} - K^{(k)})S(\hat{\mathbf{X}} - K^{(k)}) + (\alpha + \beta)(\hat{\mathbf{X}} - K^{(k)}) \leq 0,
\end{aligned}$$

we assume that $\hat{\mathbf{X}} - K^{(k)} \geq 0$, $S \leq 0$ and $\alpha + \beta < 0$. As $L^{(k)}$ is an M-matrix we conclude that $\hat{\mathbf{X}} - K^{(k+1)} \geq 0$, hence $K^{(k+1)} \leq \hat{\mathbf{X}}$.

We will prove statement (C) for $i = k + 1$ as we use (6), $P(\hat{\mathbf{X}}) > 0$ and the following equality:

$$
\begin{aligned}
& - K^{(k+1)}(\beta I_n + A - SK^{(k+1)}) - (\alpha I_{2n} + D - K^{(k+1)}S)K^{(k+1)} \\
& = -K^{(k+1)}(\beta I_n + A - SK^{(k+1)}SK^{(k)}) - (\alpha I_{2n} + D - K^{(k+1)}SK^{(k)}S)K^{(k+1)} \\
& = -K^{(k+1)}(\beta I_n + A - SK^{(k)}) + K^{(k+1)}S(K^{(k+1)} - K^{(k)}) \\
& \quad - (\alpha I_{2n} + D - K^{(k)}S)K^{(k+1)} + (K^{(k+1)} - K^{(k)})SK^{(k+1)} \\
& = Q + K^{(k)}SK^{(k)} - (\alpha + \beta)K^{(k)} + K^{(k+1)}S(K^{(k+1)} - K^{(k)}) \\
& \quad + (K^{(k+1)} - K^{(k)})SK^{(k+1)} \\
& = (K^{(k+1)} - K^{(k)})S(K^{(k+1)} - K^{(k)}) + Q - (\alpha + \beta)K^{(k)} + K^{(k+1)}SK^{(k+1)},
\end{aligned}
$$

i.e.

$$
\begin{aligned}
& -K^{(k+1)}(\beta I_n + A - SK^{(k+1)}) - (\alpha I_{2n} + D - K^{(k+1)}S)K^{(k+1)} \\
& = (K^{(k+1)} - K^{(k)})S(K^{(k+1)} - K^{(k)}) + Q - (\alpha + \beta)K^{(k)} + K^{(k+1)}SK^{(k+1)}.
\end{aligned} \tag{8}
$$

We consider the matrix

$$
\begin{aligned}
& (\hat{\mathbf{X}} - K^{(k+1)})(\beta I_n + A - SK^{(k+1)}) + (\alpha I_{2n} + D - K^{(k+1)}S)(\hat{\mathbf{X}} - K^{(k+1)}) \\
& = \hat{\mathbf{X}}(\beta I_n + A - SK^{(k+1)}) + (\alpha I_{2n} + D - K^{(k+1)}S)\hat{\mathbf{X}} \\
& \quad - K^{(k+1)}(\beta I_n + A - SK^{(k+1)}) - (\alpha I_{2n} + D - K^{(k+1)}S)K^{(k+1)},
\end{aligned}
$$

we use (8) and (7) and we obtain

$$
\begin{aligned}
= (K^{(k+1)} - K^{(k)})S(K^{(k+1)} - K^{(k)}) + (\hat{\mathbf{X}} - K^{(k+1)})S(\hat{\mathbf{X}} - K^{(k+1)}) \\
+ (\alpha + \beta)(\hat{\mathbf{X}} - K^{(k)}) \leq 0,
\end{aligned}
$$

as $\hat{\mathbf{X}} - K^{(k+1)} \geq 0$ from (B) and assume that $\alpha + \beta < 0$ and $K^{(k)} \leq K^{(k+1)}$. We conclude that

$$
L^{(k+1)} \operatorname{vec}(K^{(k+1)} - \hat{\mathbf{X}}) < 0
$$

and $L^{(k+1)}$ is a Z-matrix, and therefore also M-matrix. We will now prove property (A) for $i = k + 1$. We consider the matrix

$$
\begin{aligned}
& (K^{(k+2)} - K^{(k+1)})(\beta I_n + A - SK^{(k+1)}) + (\alpha I_{2n} + D - K^{(k+1)}S)(K^{(k+2)} - K^{(k+1)}) \\
& = K^{(k+2)}(\beta I_n + A - SK^{(k+1)}) + (\alpha I_{2n} + D - K^{(k+1)}S)K^{(k+2)} \\
& \quad - K^{(k+1)}(\beta I_n + A - SK^{(k+1)}) - (\alpha I_{2n} + D - K^{(k+1)}S)K^{(k+1)},
\end{aligned}
$$

we use (8) and iteration (6) for $i = k + 1$ than we obtain

$$
= (K^{(k+1)} - K^{(k)})S(K^{(k+1)} - K^{(k)}) + (\alpha + \beta)(K^{(k+1)} - K^{(k)}) \leq 0,
$$

as per admission $K^{(k+1)} - K^{(k)} \geq 0$, $S \leq 0$ and $\alpha + \beta < 0$. Hence $K^{(k+2)} - K^{(k+1)} \geq 0$, i.e. $K^{(k+2)} \geq K^{(k+1)}$.

The Newton sequence is monotonically increasing and bounded by $\hat{\mathbf{X}}$, hence the Newton sequence has a limit and we denote it with K, i.e. $\lim_{i\to\infty} K^{(i)} = K \geq 0$. □

2.2 Improved Iteration Method Sylvester II with Parameters (SIIsupP)

We use Sylvester II speed up iteration

$$\begin{aligned} &-(A^T - X_1^{(i)} S_1) X_1^{(i+1)} - X_1^{(i+1)} (A - S_1 X_1^{(i)} - S_2 X_2^{(i)}) = Q_1 + X_1^{(i)} S_1 X_1^{(i)}, \\ &-(A^T - X_2^{(i)} S_2) X_2^{(i+1)} - X_2^{(i+1)} (A - S_1 X_1^{(i)} - S_2 X_2^{(i)}) = Q_2 + X_2^{(i)} S_2 X_2^{(i)} \end{aligned}$$

from [2] and we produce improved iteration

$$\begin{aligned} -(\alpha I + A^T - X_1^{(i)} S_1) X_1^{(i+1)} - X_1^{(i+1)} (\beta I + A - S_1 X_1^{(i)} - S_2 X_2^{(i)}) \\ = Q_1 + X_1^{(i)} S_1 X_1^{(i)} - (\alpha + \beta) X_1^{(i)}, \end{aligned} \tag{9}$$

$$\begin{aligned} -(\alpha I + A^T - X_2^{(i)} S_2) X_2^{(i+1)} - X_2^{(i+1)} (\beta I + A - S_1 X_1^{(i)} - S_2 X_2^{(i)}) \\ = Q_2 + X_2^{(i)} S_2 X_2^{(i)} - (\alpha + \beta) X_2^{(i)}, \end{aligned} \tag{10}$$

where α and β are negative numbers, $i = 0, 1, 2, \ldots$

To study the properties of the method (9)-(10) we will introduce the following two matrix functions:

$$\begin{aligned} P_1(X_1, X_2) := &-(\alpha I + A^T - X_1 S_1) X_1 - X_1 (\beta I + A - S_1 X_1 S_2 X_2) \\ &- Q_1 - X_1 S_1 X_1 + (\alpha + \beta) X_1, \\ P_2(X_1, X_2) := &-(\alpha I + A^T - X_2 S_2) X_2 - X_2 (\beta I + A - S_1 X_1 S_2 X_2) \\ &- Q_2 - X_2 S_2 X_2 + (\alpha + \beta) X_2. \end{aligned} \tag{11}$$

We use some algebraic manipulations and we derive the identities

$$\begin{aligned} P_1(X_1, X_2) = &-(\alpha I + A^T - Y_1 S_1) X_1 - (Y_1 - X_1) S_1 X_1 - Q_1 \\ &- X_1 S_1 Y_1 - X_1 (\beta I + A - S_1 Y_1 S_2 Y_2) - X_1 S_2 (Y_2 - X_2) + (\alpha + \beta) X_1, \\ P_2(X_1, X_2) = &-(\alpha I + A^T - Y_2 S_2) X_2 - (Y_2 - X_2) S_2 X_2 - Q_2 \\ &- X_2 S_2 Y_2 - X_2 (\beta I + A - S_1 Y_1 S_2 Y_2) - X_2 S_1 (Y_1 - X_1) + (\alpha + \beta) X_2, \end{aligned} \tag{12}$$

where Y_1 and Y_2 are symmetric matrices with corresponding dimension.

For the convergence of the method we prove the following theorem:

Theorem 4 *Assume there exists two nonnegative matrices* $\hat{X}$ *and* $\mathbf{X}^{(0)}$*, such that* $0 \leq \mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$*,* $P_1(X_1^{(0)}, X_2^{(0)}) \leq 0$*,* $P_2(X_1^{(0)}, X_2^{(0)}) \leq 0$ *and* $P_1(\hat{X}_1, \hat{X}_2) \geq 0$*,* $P_2(\hat{X}_1, \hat{X}_2) \geq 0$*. The initial matrix* $\mathbf{X}^{(0)} = diag(X_1^{(0)}, X_2^{(0)})$ *we choose, so* $(-(\alpha I + A^T - X_j^{(0)} S_j))$ *for* $j = 1, 2$ *and* $(-(\beta I + A^T - X_1^{(0)} S_1 - X_2^{(0)} S_2))$ *are M-matrices, where* α *and* β *are negative numbers, and* I *is* $n \times n$ *unit matrix. Then, the matrix sequences* $\{X_1^{(i)}, X_2^{(i)}\}_{i=0}^{\infty}$*, defined by (9)-(10), satisfy:*

(i) $\mathbf{X}^{(i+1)} \geq \mathbf{X}^{(i)}$*,* $i = 0, 1, 2, \ldots$*;*

(ii) $\mathbf{X}^{(i+1)} \leq \hat{\mathbf{X}}$*,* $i = 0, 1, 2, \ldots$*;*

(iii) $(-(\alpha I + A^T - X_j^{(i+1)} S_j))$ *for* $j = 1, 2$ *and* $(-(\beta I + A^T - X_1^{(i+1)} S_1 - X_2^{(i+1)} S_2))$ *are M-matrices for* $i = 0, 1, 2, \ldots$*;*

(iv) The matrix sequences $\{X_1^{(i)}, X_2^{(i)}\}_{i=0}^{\infty}$ *converge to the nonnegative minimal solution* $\tilde{\mathbf{X}} = (\tilde{X_1}, \tilde{X_2})$ *to the Riccati equation (1), where* $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$*.*

Proof: We apply (11) to $P_1(X_1, X_2)$ for $X_j = \hat{X}_j$, $j = 1, 2$ and we obtain

$$A^T\hat{X}_1 + \hat{X}_1 A = -P_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 \hat{X}_1 + \hat{X}_1 S_2 \hat{X}_2 - Q_1. \tag{13}$$

We denote

$$L_j^{(i)} = -[(\beta I + A^T - X_1^{(i)} S_1 - X_2^{(i)} S_2) \otimes I_n + I_n \otimes (\alpha I + A^T - X_j^{(i)} S_j)], \tag{14}$$

for $j = 1, 2$, $i = 0, 1, 2....$ From iterative formula (9) we express

$$\begin{aligned} -A^T X_1^{(i+1)} - X_1^{(i+1)} A = (\alpha+\beta) X_1^{(i+1)} - X_1^{(i)} S_1 X_1^{(i+1)} - X_1^{(i+1)} S_1 X_1^{(i)} \\ - X_1^{(i+1)} S_2 X_2^{(i)} + Q_1 + X_1^{(i)} S_1 X_1^{(i)} - (\alpha+\beta) X_1^{(i)}. \end{aligned} \tag{15}$$

Let there exists two nonnegative matrices $\hat{\mathbf{X}}$ and $\mathbf{X^{(0)}}$, such that $0 \le \mathbf{X^{(0)}} \le \hat{\mathbf{X}}$, $P_j(X_1^{(0)}, X_2^{(0)}) \le 0$ and $P_j(\hat{X}_1, \hat{X}_2) \ge 0$ for $j = 1, 2$. From

$$\begin{aligned} P_1(X_1^{(0)}, X_2^{(0)}) = &-(\alpha I + A^T - X_1^{(0)} S_1) X_1^{(0)} - X_1^{(0)} (\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) \\ &- Q_1 - X_1^{(0)} S_1 X_1^{(0)} + (\alpha+\beta) X_1^{(0)} \le 0 \end{aligned}$$

that $P_1(X_1^{(0)}, X_2^{(0)}) = Q_1^{(0)} \le 0$. The initial matrix $\mathbf{X}^{(0)} = diag(X_1^{(0)}, X_2^{(0)})$ we choose, so $(-(\alpha I + A^T - X_j^{(0)} S_j))$ for $j = 1, 2$ and $(-(\beta I + A^T - X_1^{(0)} S_1 - X_2^{(0)} S_2))$ are M-matrices. We will prove that $X_1^{(1)} - X_1^{(0)}$ is a nonnegative matrix. We consider the difference between iterative formula (9) at $i = 0$ and $P_1(X_1^{(0)}, X_2^{(0)})$:

$$\begin{aligned} 0 - &P_1(X_1^{(0)}, X_2^{(0)}) \\ &= -(\alpha I + A^T - X_1^{(0)} S_1) X_1^{(1)} - X_1^{(1)} (\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) \\ &\quad - Q_1 - X_1^{(0)} S_1 X_1^{(0)} + (\alpha+\beta) X_1^{(0)} \\ &\quad - [-(\alpha I + A^T - X_1^{(0)} S_1) X_1^{(0)} - X_1^{(0)} (\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) \\ &\quad - Q_1 - X_1^{(0)} S_1 X_1^{(0)} + (\alpha+\beta) X_1^{(0)}], \end{aligned}$$

i.e.

$$\begin{aligned} -Q_1^{(0)} = &-(\alpha I + A^T - X_1^{(0)} S_1)(X_1^{(1)} - X_1^{(0)}) \\ &- (X_1^{(1)} - X_1^{(0)})(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}). \end{aligned}$$

This is a linear matrix equation of Sylvester against the unknown matrix $(X_1^{(1)} - X_1^{(0)})$. We find the solution by solving the linear system of equations:

$$\begin{aligned} &[(-(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)})^T) \otimes I_n \\ &\qquad + I_n \otimes (-(\alpha I + A^T - X_1^{(0)} S_1))] \operatorname{vec}(X_1^{(1)} - X_1^{(0)}) = -\operatorname{vec} Q_1^{(0)} \end{aligned}$$

or

$$\begin{aligned} &[(-(\beta I + A^T - X_1^{(0)} S_1 - X_2^{(0)} S_2)) \otimes I_n \\ &\qquad + I_n \otimes (-(\alpha I + A^T - X_1^{(0)} S_1))] \operatorname{vec}(X_1^{(1)} - X_1^{(0)}) = -\operatorname{vec} Q_1^{(0)}. \end{aligned}$$

Therefore, using (14) at $j = 1$ and $i = 0$ the equation yields the form

$$L_1^{(0)} \operatorname{vec}(X_1^{(1)} - X_1^{(0)}) = -\operatorname{vec} Q_1^{(0)}.$$

The matrix $\mathbf{X}^{(0)} = diag\,(X_1^{(0)}, X_2^{(0)})$ is a nonnegative matrix and $(-(\alpha I + A^T - X_1^{(0)} S_1))$ and $(-(\beta I + A^T - X_1^{(0)} S_1 - X_2^{(0)} S_2))$ are M-matrices. Then we obtain that the matrix $L_1^{(0)}$ is an M-matrix, and this means that $(L_1^{(0)})^{-1} \geq 0$ and since $-vec\,Q_1^{(0)} \geq 0$, then

$$vec\,(X_1^{(1)} - X_1^{(0)}) = -(L_1^{(0)})^{-1} \operatorname{vec} Q_1^{(0)} \geq 0,$$

which means $X_1^{(1)} - X_1^{(0)} \geq 0$, or $X_1^{(1)} \geq X_1^{(0)}$. The inequality $X_2^{(1)} \geq X_2^{(0)}$ is proved analogously. Hence

$$\mathbf{X}^{(1)} \geq \mathbf{X}^{(0)}.$$

Thus we have proven property (i) at $i = 0$. We will prove property (ii) at $i = 0$. We know that the matrices $(-(\alpha I + A^T - X_1^{(0)} S_1))$ and $(-(\beta I + A^T - X_1^{(0)} S_1 - X_2^{(0)} S_2))$ are M-matrices. The aim is to prove that the matrix $\hat{X}_1 - X_1^{(1)}$ is nonnegative matrix. We form the matrix

$$(\alpha I + A^T - X_1^{(0)} S_1)(\hat{X}_1 - X_1^{(1)}) + (\hat{X}_1 - X_1^{(1)})(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) \qquad (16)$$

and we calculate for it

$$\begin{aligned}
&= (\alpha I + A^T - X_1^{(0)} S_1)\hat{X}_1 - (\alpha I + A^T - X_1^{(0)} S_1) X_1^{(1)} \\
&\quad + \hat{X}_1(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) - X_1^{(1)}(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) \\
&= \alpha \hat{X}_1 + A^T \hat{X}_1 - X_1^{(0)} S_1 \hat{X}_1 + \beta \hat{X}_1 + \hat{X}_1 A - \hat{X}_1 S_1 X_1^{(0)}) - \hat{X}_1 S_2 X_2^{(0)} \\
&\quad + Q_1 + X_1^{(0)} S_1 X_1^{(0)}) - (\alpha + \beta) X_1^{(0)}.
\end{aligned}$$

Therefore, using (13) for the matrix (16) we obtain

$$\begin{aligned}
&= -P_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 \hat{X}_1 + \hat{X}_1 S_2 \hat{X}_2 - Q_1 + \alpha \hat{X}_1 - X_1^{(0)} S_1 \hat{X}_1 \\
&\quad + \beta \hat{X}_1 - \hat{X}_1 S_1 X_1^{(0)} - \hat{X}_1 S_2 X_2^{(0)} + Q_1 + X_1^{(0)} S_1 X_1^{(0)} - (\alpha + \beta) X_1^{(0)} \\
&= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(0)}) S_1 \hat{X}_1 - (\hat{X}_1 - X_1^{(0)}) S_1 X_1^{(0)} \\
&\quad + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(0)}) + (\alpha + \beta)(\hat{X}_1 - X_1^{(0)}) \\
&= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(0)}) S_1 (\hat{X}_1 - X_1^{(0)}) + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(0)}) \\
&\quad + (\alpha + \beta)(\hat{X}_1 - X_1^{(0)}) \\
&= N_1^{(1)}.
\end{aligned}$$

We assume that $P_1(\hat{X}_1, \hat{X}_2) \geq 0$, S_1 and $S_2 \leq 0$, $\hat{X}_1$ and $\hat{X}_2 \geq 0$, $\alpha + \beta < 0$, and $\hat{X}_j \geq X_j^{(0)}$ for $j = 1, 2$. Hence, the matrix $N_1^{(1)}$ is nonpositive, $N_1^{(1)} \leq 0$. Thus the matrix (16) sets the following matrix equation

$$-(\alpha I + A^T - X_1^{(0)} S_1)(\hat{X}_1 - X_1^{(1)}) - (\hat{X}_1 - X_1^{(1)})(\beta I + A - S_1 X_1^{(0)} - S_2 X_2^{(0)}) = -N_1^{(1)} \geq 0$$

or

$$L_1^{(0)}\mathrm{vec}\,(\hat{X}_1 - X_1^{(1)}) = -\mathrm{vec}\,N_1^{(1)}.$$

Hence, $\hat{X}_1 - X_1^{(1)} \geq 0$, or $\hat{X}_1 \geq X_1^{(1)}$. Analogously, we will prove the inequality $\hat{X}_2 \geq X_2^{(1)}$. Therefore

$$\hat{\mathbf{X}} \geq \mathbf{X}^{(1)}.$$

We proved property (ii) at $i = 0$.

We will prove property (iii) at $i = 0$. We consider the matrix

$$(\alpha I + A^T - X_1^{(1)}S_1)(\hat{X}_1 - X_1^{(1)}) + (\hat{X}_1 - X_1^{(1)})(\beta I + A - S_1X_1^{(1)} - S_2X_2^{(1)}). \tag{17}$$

Using (13), and (15) at $i = 0$ we obtain

$$\begin{aligned}
&= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(1)})S_1\hat{X}_1 + \hat{X}_1S_2(\hat{X}_2 - X_2^{(1)}) - Q_1 - \hat{X}_1S_1X_1^{(1)} \\
&\quad + (\alpha+\beta)\hat{X}_1 + (X_1^{(1)} - X_1^{(0)})S_1X_1^{(1)} - (X_1^{(1)} - X_1^{(0)})S_1X_1^{(0)} + Q_1 \\
&\quad + X_1^{(1)}S_1X_1^{(1)} - (\alpha+\beta)X_1^{(0)} + X_1^{(1)}S_2(X_2^{(1)} - X_2^{(0)}) \\
&= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(1)})S_1\hat{X}_1 + \hat{X}_1S_2(\hat{X}_2 - X_2^{(1)}) - (\hat{X}_1 - X_1^{(1)})S_1X_1^{(1)} \\
&\quad + (\alpha+\beta)(\hat{X}_1 - X_1^{(0)}) + (X_1^{(1)} - X_1^{(0)})S_1(X_1^{(1)} - X_1^{(0)}) + X_1^{(1)}S_2(X_2^{(1)} - X_2^{(0)}) \\
&= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(1)})S_1(\hat{X}_1 - X_1^{(1)}) + \hat{X}_1S_2(\hat{X}_2 - X_2^{(1)}) \\
&\quad + (\alpha+\beta)(\hat{X}_1 - X_1^{(0)}) + (X_1^{(1)} - X_1^{(0)})S_1(X_1^{(1)} - X_1^{(0)}) + X_1^{(1)}S_2(X_2^{(1)} - X_2^{(0)}) \\
&= R_1^{(1)}.
\end{aligned}$$

Hence,

$$(\alpha I + A^T - X_1^{(1)}S_1)(\hat{X}_1 - X_1^{(1)}) + (\hat{X}_1 - X_1^{(1)})(\beta I + A - S_1X_1^{(1)} - S_2X_2^{(1)}) = R_1^{(1)}.$$

Since $\hat{X}_j \geq 0$, $P_j(\hat{X}_1, \hat{X}_2) \geq 0$, $\hat{X}_j - X_j^{(1)} \geq 0$ for $j = 1, 2$, and $X_j^{(1)} \geq 0$, $\hat{X}_j - X_j^{(0)} \geq 0$, $X_j^{(1)} - X_j^{(0)} \geq 0$, $S_j \leq 0$ for $j = 1, 2$, $\alpha + \beta < 0$, then $R_1^{(1)} \leq 0$.

We consider the matrix $\hat{X}_1 - X_1^{(1)}$ as a non-negative solution to the Sylvester equation in the form

$$-(\alpha I + A^T - X_1^{(1)}S_1)(\hat{X}_1 - X_1^{(1)}) - (\hat{X}_1 - X_1^{(1)})(\beta I + A - S_1X_1^{(1)} - S_2X_2^{(1)}) = -R_1^{(1)}.$$

We present the solution:

$$0 \leq \mathrm{vec}\,(\hat{X}_1 - X_1^{(1)}) = -(L_1^{(1)})^{-1}\mathrm{vec}\,R_1^{(1)}, (-\mathrm{vec}\,R_1^{(1)} \geq 0).$$

Therefore $(L_1^{(1)})^{-1} \geq 0$, i.e. the matrices $(-(\alpha I + A^T - X_1^{(1)}S_1))$ and $(-(\beta I + A^T - X_1^{(1)}S_1 - X_2^{(1)}S_2))$ are M-matrices.

Analogously, we will prove that $(L_2^{(1)})^{-1}$ is a nonnegative matrix, i.e. the matrices $(-(\alpha I + A^T - X_2^{(1)}S_2))$ and $(-(\beta I + A^T - X_1^{(1)}S_1 - X_2^{(1)}S_2))$ are M-matrices.

We proved property (iii) at $i = 0$.
Assume that the statements: $\mathbf{X}^{(i+1)} \geq \mathbf{X}^{(i)} \geq 0$, $\mathbf{X}^{(i+1)} \leq \hat{\mathbf{X}}$, $(-(\alpha I + A^T - X_j^{(i+1)} S_j))$ for $j = 1, 2$ and $(-(\beta I + A^T - X_1^{(i+1)} S_1 - X_2^{(i+1)} S_2))$ are M-matrices, are fulfilled for $i = k > 0$.

We will prove the statements for $i = k + 1$.
(a_{k+1}) $\mathbf{X}^{(k+2)} \geq \mathbf{X}^{(k+1)} \geq 0$;
(b_{k+1}) $\mathbf{X}^{(k+2)} \leq \hat{\mathbf{X}}$;
(c_{k+1}) $(-(\alpha I + A^T - X_j^{(k+2)} S_j))$ for $j = 1, 2$ and $(-(\beta I + A^T - X_1^{(k+2)} S_1 - X_2^{(k+2)} S_2))$ are M-matrices.
We will prove the statement (a_{k+1}). We calculate $X_1^{(k+2)}$, $X_2^{(k+2)}$ via (9)-(10). Since matrix coefficients in (9)-(10) at $i = k+1$ are M-matrices, then the solutions found are non-negative. At first we will prove that $P_j(X_1^{(k+1)}, X_2^{(k+1)})$ for $j = 1, 2$ are non-positive matrices. We apply (12) to $P_1(X_1^{(k+1)}, X_2^{(k+1)})$ and obtain

$$\begin{aligned}
&P_1(X_1^{(k+1)}, X_2^{(k+1)}) = \\
&= -(\alpha I + A^T - X_1^{(k)} S_1) X_1^{(k+1)} - (X_1^{(k)} - X_1^{(k+1)}) S_1 X_1^{(k+1)} - Q_1 - X_1^{(k+1)} S_1 X_1^{(k)} \\
&\quad - X_1^{(k+1)} (\beta I + A - S_1 X_1^{(k)} S_2 X_2^{(k)}) - X_1^{(k+1)} S_2 (X_2^{(k)} - X_2^{(k+1)}) + (\alpha + \beta) X_1^{(k+1)}.
\end{aligned}$$

Using (15) at $i = k$ we write

$$\begin{aligned}
P_1(X_1^{(k+1)}, X_2^{(k+1)}) &= Q_1 + X_1^{(k)} S_1 X_1^{(k)} - (\alpha + \beta) X_1^{(k)} - (X_1^{(k)} - X_1^{(k+1)}) S_1 X_1^{(k+1)} \\
&\quad - Q_1 - X_1^{(k+1)} S_1 X_1^{(k)} - X_1^{(k+1)} S_2 (X_2^{(k)} - X_2^{(k+1)}) + (\alpha + \beta) X_1^{(k+1)} \\
&= -(X_1^{(k+1)} - X_1^{(k)}) S_1 X_1^{(k)} + (X_1^{(k+1)} - X_1^{(k)}) S_1 X_1^{(k+1)} \\
&\quad + X_1^{(k+1)} S_2 (X_2^{(k+1)} - X_2^{(k)}) + (\alpha + \beta)(X_1^{(k+1)} - X_1^{(k)}) \\
&= (X_1^{(k+1)} - X_1^{(k)}) S_1 (X_1^{(k+1)} - X_1^{(k)}) + X_1^{(k+1)} S_2 (X_2^{(k+1)} - X_2^{(k)}) \\
&\quad + (\alpha + \beta)(X_1^{(k+1)} - X_1^{(k)}) \leq 0,
\end{aligned}$$

i.e. function $P_1(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0$, since $X_1^{(k+1)} \geq X_1^{(k)}$, $X_2^{(k+1)} \geq X_2^{(k)}$, $X_1^{(k+1)} \geq 0$ and $\alpha + \beta < 0$. We obtain that $P_1(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0$ and analogously hence that $P_2(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0$.

We consider the difference between iterative formula (9) at $i = k+1$ and $P_1(X_1^{(k+1)}, X_2^{(k+1)})$ represented by (11):

$$\begin{aligned}
&0 - P_1(X_1^{(k+1)}, X_2^{(k+1)}) \\
&= -(\alpha I + A^T - X_1^{(k+1)} S_1) X_1^{(k+2)} - X_1^{(k+2)} (\beta I + A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) \\
&\quad - Q_1 - X_1^{(k+1)} S_1 X_1^{(k+1)} + (\alpha + \beta) X_1^{(k+1)} \\
&\quad - [-(\alpha I + A^T - X_1^{(k+1)} S_1) X_1^{(k+1)} - X_1^{(k+1)} (\beta I + A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) \\
&\quad - Q_1 - X_1^{(k+1)} S_1 X_1^{(k+1)} + (\alpha + \beta) X_1^{(k+1)}].
\end{aligned}$$

We denote $Q_1^{(k+1)} := P_1(X_1^{(k+1)}, X_2^{(k+1)})$ and obtain

$$\begin{aligned} Q_1^{(k+1)} &= -(\alpha I + A^T - X_1^{(k+1)} S_1)(X_1^{(k+2)} - X_1^{(k+1)}) \\ &\quad - (X_1^{(k+2)} - X_1^{(k+1)})(\beta I + A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}), \end{aligned}$$

this is a linear matrix Sylvester equation to unknown matrix $(X_1^{(k+2)} - X_1^{(k+1)})$. We find the solution by solving the linear system of equations:

$$\begin{aligned} &[(-(\beta I + A^T - X_1^{(k+1)} S_1 - X_2^{(k+1)} S_2)) \otimes I_n \\ &\quad + I_n \otimes (-(\alpha I + A^T - X_1^{(k+1)} S_1))] \mathrm{vec}\,(X_1^{(k+2)} - X_1^{(k+1)}) = -\mathrm{vec}\, Q_1^{(k+1)}. \end{aligned}$$

Using (14) for $j = 1$, $i = k + 1$ we rewrite the equation

$$L_1^{(k+1)} \mathrm{vec}\,(X_1^{(k+2)} - X_1^{(k+1)}) = -\mathrm{vec}\, Q_1^{(k+1)}.$$

The matrices $X_1^{(k+1)}$, $X_2^{(k+1)}$ are nonnegative and relevant matrices $(-(\alpha I + A^T - X_1^{(k+1)} S_1))$ and $(-(\beta I + A^T - X_1^{(k+1)} S_1 - X_2^{(k+1)} S_2))$ are M-matrices. Thus the matrix $L_1^{(k+1)}$ is an M-matrix and that means that $(L_1^{(k+1)})^{-1} \geq 0$. And since $-\mathrm{vec}\, Q_1^{(k+1)} \geq 0$, then $\mathrm{vec}\,(X_1^{(k+2)} - X_1^{(k+1)}) = -(L_1^{(k+1)})^{-1} vec\, Q_1^{(k+1)} \geq 0$, which means $X_1^{(k+2)} - X_1^{(k+1)} \geq 0$, or $X_1^{(k+2)} \geq X_1^{(k+1)}$. Analogously, the inequality $X_2^{(k+2)} \geq X_2^{(k+1)}$ can be proved. Therefore

$$\mathbf{X}^{(k+2)} \geq \mathbf{X}^{(k+1)}.$$

With this we proved a property (a_{k+1}).
We will prove property (b_{k+1}) at $i = k + 1$, i.e. $\hat{X}_1 \geq X_1^{(k+2)}$ and $\hat{X}_2 \geq X_2^{(k+2)}$. We know that the matrices $(-(\alpha I + A^T - X_1^{(k+1)} S_1))$ and $(-(\beta I + A^T - X_1^{(k+1)} S_1 - X_2^{(k+1)} S_2))$ are M-matrices. We form the matrix

$$(\alpha I + A^T - X_1^{(k+1)} S_1)(\hat{X}_1 - X_1^{(k+2)}) + (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}). \quad (18)$$

The aim is to prove that the matrix $\hat{X}_1 - X_1^{(k+2)}$ is nonnegative. Using iteration (9) at $i = k + 1$ and (13) for the matrix (18) we obtain

$$\begin{aligned} &= -P_1(\hat{X}_1, \hat{X}_2) + (\hat{X}_1 - X_1^{(k+1)}) S_1 (\hat{X}_1 - X_1^{(k+1)}) + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(k+1)}) \\ &\qquad + (\alpha + \beta)(\hat{X}_1 - X_1^{(k+1)}) = N_1^{(k+2)}. \end{aligned}$$

Then the matrix (18) sets the following matrix equation

$$\begin{aligned} &- (\alpha I + A^T - X_1^{(k+1)} S_1)(\hat{X}_1 - X_1^{(k+2)}) \\ &\qquad - (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+1)} - S_2 X_2^{(k+1)}) = -N_1^{(k+2)} \geq 0. \end{aligned}$$

Using (14) for $j = 1$, $i = k + 1$ we rewrite the equation

$$L_1^{(k+1)} \mathrm{vec}\,(\hat{X}_1 - X_1^{(k+2)}) = -\mathrm{vec}\, N_1^{(k+2)}.$$

Since $P_1(\hat{X}_1, \hat{X}_2) \geq 0$, S_1 and S_2 are nonpositive, $\hat{X}_1$ and $\hat{X}_2$ are nonnegative, $\hat{X}_j \geq X_j^{(k+1)}$ for $j = 1, 2$, $\alpha + \beta < 0$. Hence the matrix $N_1^{(k+2)}$ is a nonpositive, $N_1^{(k+2)} \leq 0$. Hence $\hat{X}_1 - X_1^{(k+2)} \geq 0$, or $\hat{X}_1 \geq X_1^{(k+2)}$. Analogously, the inequality $\hat{X}_2 \geq X_2^{(k+2)}$ can be proved. Therefore

$$\hat{\mathbf{X}} \geq \mathbf{X}^{(k+2)}.$$

With this we proved a property (b_{k+1}).
We will prove property (c_{k+1}) at $i = k+1$. We consider the matrix

$$(\alpha I + A^T - X_1^{(k+2)} S_1)(\hat{X}_1 - X_1^{(k+2)}) + (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+2)} - S_2 X_2^{(k+2)}). \quad (19)$$

Using iteration (9) at $i = k+1$ and (13) we obtain

$$\begin{aligned}
&(\alpha I + A^T - X_1^{(k+2)} S_1)(\hat{X}_1 - X_1^{(k+2)}) \\
&\quad + (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+2)} - S_2 X_2^{(k+2)}) \\
&= (\alpha + \beta)\hat{X}_1 - X_1^{(k+2)} S_1 \hat{X}_1 - \hat{X}_1 S_1 X_1^{(k+2)} - \hat{X}_1 S_2 X_2^{(k+2)} \\
&\quad - P_1(\hat{X}_1, \hat{X}_2) + \hat{X}_1 S_1 \hat{X}_1 + \hat{X}_1 S_2 \hat{X}_2 - Q_1 \\
&\quad + X_1^{(k+2)} S_1 X_1^{(k+2)} + X_1^{(k+2)} S_1 X_1^{(k+2)} + X_1^{(k+2)} S_2 X_2^{(k+2)} - X_1^{(k+1)} S_1 X_1^{(k+2)} \\
&\quad - X_1^{(k+2)} S_1 X_1^{(k+1)} - X_1^{(k+2)} S_2 X_2^{(k+1)} + Q_1 + X_1^{(k+1)} S_1 X_1^{(k+1)} - (\alpha + \beta) X_1^{(k+1)} \\
&= (\alpha + \beta)(\hat{X}_1 - X_1^{(k+1)}) + (\hat{X}_1 - X_1^{(k+2)}) S_1 \hat{X}_1 - \hat{X}_1 S_1 X_1^{(k+2)} \\
&\quad + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(k+2)}) - P_1(\hat{X}_1, \hat{X}_2) + (X_1^{(k+2)} - X_1^{(k+1)}) S_1 (X_1^{(k+2)} - X_1^{(k+1)}) \\
&\quad + X_1^{(k+2)} S_2 (X_2^{(k+2)} - X_2^{(k+1)}) + X_1^{(k+2)} S_1 X_1^{(k+2)} \\
&= (\alpha + \beta)(\hat{X}_1 - X_1^{(k+1)}) + (\hat{X}_1 - X_1^{(k+2)}) S_1 (\hat{X}_1 - X_1^{(k+2)}) \\
&\quad + \hat{X}_1 S_2 (\hat{X}_2 - X_2^{(k+2)}) - P_1(\hat{X}_1, \hat{X}_2) + (X_1^{(k+2)} - X_1^{(k+1)}) S_1 (X_1^{(k+2)} - X_1^{(k+1)}) \\
&\quad + X_1^{(k+2)} S_2 (X_2^{(k+2)} - X_2^{(k+1)}) = R_1^{(k+2)}.
\end{aligned}$$

That's how we came to equality

$$\begin{aligned}
&(\alpha I + A^T - X_1^{(k+2)} S_1)(\hat{X}_1 - X_1^{(k+2)}) \\
&\qquad\qquad + (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+2)} - S_2 X_2^{(k+2)}) = R_1^{(k+2)}.
\end{aligned}$$

Since $\hat{X}_j \geq 0$, $P_j(\hat{X}_1, \hat{X}_2) \geq 0$, $\hat{X}_j - X_j^{(k+2)} \geq 0$ for $j = 1, 2$, $X_j^{(k+2)} \geq 0$, $S_j \leq 0$ for $j = 1, 2$, $\alpha + \beta < 0$, and we have already proven $X_j^{(k+2)} - X_j^{(k+1)} \geq 0$ for $j = 1, 2$, then $R_1^{(k+2)} \leq 0$. Consider the matrix $\hat{X}_1 - X_1^{(k+2)}$ as a nonnegative solution to the Sylvester equation

$$\begin{aligned}
&-(\alpha I + A^T - X_1^{(k+2)} S_1)(\hat{X}_1 - X_1^{(k+2)}) \\
&\qquad\qquad - (\hat{X}_1 - X_1^{(k+2)})(\beta I + A - S_1 X_1^{(k+2)} - S_2 X_2^{(k+2)}) = -R_1^{(k+2)}.
\end{aligned}$$

We present the solution:

$$0 \leq \mathrm{vec}\,(\hat{X}_1 - X_1^{(k+2)}) = -(L_1^{(k+2)})^{-1}\,\mathrm{vec}\,R_1^{(k+2)}, (-\mathrm{vec}\,R_1^{(k+2)} \geq 0).$$

This means that $(L_1^{(k+2)})^{-1}$ is a nonnegative matrix, i.e. the matrices $(-(\alpha I + A^T - X_1^{(k+2)} S_1))$ and $(-(\beta I + A^T - X_1^{(k+2)} S_1 - X_2^{(k+2)} S_2))$ are M-matrices.
Analogously, it is proved that $(L_2^{(k+2)})^{-1}$ is a nonnegative matrix, i.e. the matrices $(-(\alpha I + A^T - X_2^{(k+2)} S_2))$ and $(-(\beta I + A^T - X_1^{(k+2)} S_1 - X_2^{(k+2)} S_2))$ are M-matrices. With this we proved a property (c_{k+1}).
Hence, the induction process has been completed.
Consequently the matrix sequence $\{X_1^{(i)}, X_2^{(i)}\}_{i=0}^{\infty}$ is monotonically increasing and bounded above by $\hat{X}_1$, $\hat{X}_2$. We denote $\lim_{i\to\infty}(X_1^{(i)}, X_2^{(i)}) = (\tilde{X}_1, \tilde{X}_2)$. $(\tilde{X}_1, \tilde{X}_2)$ is a solution of the equation (1) with the property $\tilde{X}_j \leq \hat{X}_j$ for $j = 1, 2$ and matrices $I_n \otimes (-(\alpha I + A - S_1\tilde{X}_1)^T) + (-(\beta I + A - S_1\tilde{X}_1 - S_2\tilde{X}_2)^T) \otimes I_n$ and $I_n \otimes (-(\alpha I + A - S_2\tilde{X}_2)^T) + (-(\beta I + A - S_1\tilde{X}_1 - S_2\tilde{X}_2)^T) \otimes I_n$ are M-matrices.
The proof is complete. □

3 Numerical Experiments

We will now describe numerical experiments for calculating the stabilizing solution of the studied equation (1). To find the solution, we will use the described iteration algorithms the Newton Method (NM), the Newton Method with Parameters (NMP) and the Linearized Newton Method (LNM). As the matrix coefficients are formed by Matlab's random number function *randn* (R2009b), we will do a series of experiments. We will fix the dimension of n and we will seek the solution by any of the methods described. As a result of the experiments for each value of n and each method we will note the following parameters: "Max It" the highest number of iterations, "Av It" average number of iterations and "CPU time" processor time.

Example 1. The matrix coefficients are:
α=-0.001;
β=-0.002;
γ=-4;
for k=1:200
A=abs(randn(n))/99;
s=max(abs(eig(A)))+4.5;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1 = zeros(n,1); B_1(1)=abs(randn(1,1))/5;
B_2=eye(n,n); B_2(n,n)=n/3;
Q_1=zeros(n,n); Q_1(1,1)=n/5; Q_1(n,n)=1.5;
R_{11}=-1;
$Q_2 = Q_1$;
R_{22}=-eye(n,n); R_{22}(1,1)=-40; R_{22}(n,n)=-30;
To conduct the experiment, we change the dimension $n = 12$ and 25. We have 200 runs for

Table 1.

	NM (2)			NMP (6)			LNM (4) - (5)		
n	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU
12	3	3	0.705s	4	4	0.785s	5	5.00	0.534s
25	4	4	2.246s	4	4	1.618s	6	5.02	1.220s

Table 2.

	NM (2)			NMP (6)			LNM (4) - (5)		
n	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU
12	3	3	0.894s	5	4.09	0.952s	4	4.00	0.583s
25	4	4	2.532s	5	5.00	2.445s	5	4.25	1.424s

each value of n, repeat this 10 times and we calculate average values for "Max It", "Av It" and "CPU time". The results are described in Table 1.

For matrices of larger size the Newton Method with Parameters is faster than the Newton Method. The Linearized Newton Method is faster than the Newton Method and the Newton Method with Parameters at different sizes of the participating matrix coefficients.

Example 2. The matrix coefficients are:
α=-0.001;
β=-0.003;
γ=-1.5;
for k=1:250
A=abs(randn(n))/100;
s=max(abs(eig(A)))+1.5;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1=abs(randn(n,1))/10;
B_2=eye(n,n); B_2(n,n)=n/5; B_2(1,1)=n/10;
Q_1=zeros(n,n); Q_1(1,1)=n/5; Q_1(n,n)=1/n;
$Q_2 = Q_1$;
R_{11}=-1.5;
R_{22}=-eye(n,n); R_{22}(1,1)=-50; R_{22}(n,n)=-20;

To conduct the experiment, we change the dimension $n = 12$ and 25. We have 250 runs for each value of n, repeat this 10 times and we calculate average values for "Max It", "Av It" and "CPU time". The results are described in Table 2.

The Linearized Newton Method is faster than the Newton Method and the Newton Method with Parameters at different matrix coefficient dimensions.

Example 3. The matrix coefficients are:
α=-0.001;
β=-0.002;
γ=-5;

Table 3a.

	NM (2)			NMP (6)		
n	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU
25	3	3	0.64s	4	4.0	0.88s
50	4	4	3.17s	5	4.5	3.77s
75	4	4	8.93s	5	5.0	11.39s
100	4	4	23.14s	5	5.0	28.45s
125	5	4	56.09s	5	5.0	73.65s

Table 3b.

	LNM (4) - (5)			SIIsupP (9) - (10)		
n	*maxIt*	*avIt*	CPU	*maxIt*	*avIt*	CPU
25	5	5.0	0.64s	4	4.0	1.62s
50	5	5.0	2.07s	5	5.0	5.70s
75	6	5.4	4.52s	6	5.2	13.74s
100	7	6.0	9.63s	7	5.9	39.47s
125	7	7.0	26.02s	8	6.4	93.55s

for k=1:100
A=abs(randn(n))/50;
s=max(abs(eig(A)))+4.5;
for i=1:n, A(i,i)=-(A(i,i))-s; end
B_1=abs(randn(n,1))/10;
B_2=eye(n,n); B_2(n,n)=n/5;
Q_1=eye(n,n); Q_1(1,n)=n; Q_1(n,1)=1;
Q_2=0.5*eye(n,n);
R_{11}=-1.5;
R_{22}=-10*eye(n,n); R_{22}(1,1)=-30; R_{22}(n,n)=-20;

We change the dimension n: $n = 25, 50, 75, 100$ and 125. We have 100 runs for each value of n with each of four methods the Newton Method, the Newton Method with Parameters, the Linearized Newton Method and the Sylvester II speed up Method with Parameters. The results are described in Table 4.

The linearized Newton method remains the fastest.

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